

The Comparison Matrix Between the Two First-Cut Factorizations

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Verdict: scope-restriction (resolved).

1 The comparison step

Let $\mathcal{U}$ be a matrix product operator tensor with physical dimension d and bond dimension D , let M_1 and M_2 be its two matrix flattenings, and let r and ℓ be their ranks. Section III of Ref. [CPGSV17] constructs factorizations $M_1 = \tilde{X}_1 \tilde{Y}_1$ and $M_2 = X_2 Y_2$ from singular value decompositions, together with the auxiliary factors $\tilde{Z}_1$ and Z_2 that invert $\tilde{Y}_1$ and Y_2 on the right. Writing ρ for the positive diagonal fixed point of the normalized transfer map and $W_\rho = \text{Id}_d \otimes \rho$ for the associated weight, the printed normalizations are

$$X_2^\dagger X_2 = \text{Id}_\ell, \quad (1)$$

$$\tilde{X}_1^\dagger W_\rho \tilde{X}_1 = \text{Id}_r. \quad (2)$$

The second factorization is a plain isometry, whereas the first is an isometry only against the weight W_ρ .¹

The appendix of Ref. [FRBC25] treats a tensor whose adjoint is a unitary virtual gauge conjugate of itself, $\mathcal{U}_{ij}^\dagger = T^\dagger \mathcal{U}_{ij} T$, and builds a second factorization $M_1 = X_1 Y_1$ out of the second-cut factors,

$$X_1 = A Y_2^\dagger, \quad (3)$$

$$Y_1 = X_2^\dagger B, \quad (4)$$

where A and B are the virtual dressings built from T . It then compares the two factorizations of M_1 through a single invertible matrix K , printed as

$$K = \tilde{X}_1^\dagger X_1 = \tilde{Y}_1 Y_1^\dagger, \quad (5)$$

$$\tilde{Y}_1 = K Y_1, \quad (6)$$

$$\tilde{X}_1 = X_1 K^\dagger. \quad (7)$$

¹Source traceability: `Papers/1703.09188/paper_v2.tex`, lines 480–500. The declared auxiliary-label build for this archived source fails on the title's malformed accent command, so the printed section designation is used.

This note concerns (5) and (7).² The two deviate from the formal statement in different ways. The first printed expression for K needs the weight W_ρ inserted; that is a local correction, and the formal development applies it, so nothing about it remains open. The second names $K^\dagger$ as the inverse of K , hence asserts that K is unitary. The formal development first exhibited only a separately constructed two-sided inverse, which made its statement a restriction of the printed one and fixed the classification of this note; that restriction has since been removed, as recorded in the verdict.

2 The comparison matrix is determined, and carries the weight

Suppose $M_1 = X_1 Y_1 = \tilde{X}_1 \tilde{Y}_1$, that $\tilde{L}$ is a left inverse of $\tilde{X}_1$, and that R is a right inverse of Y_1 . Put $K = \tilde{L} X_1$. Then

$$\tilde{Y}_1 = \tilde{L} M_1 = \tilde{L} X_1 Y_1 = K Y_1, \quad (8)$$

$$X_1 = M_1 R = \tilde{X}_1 \tilde{Y}_1 R = \tilde{X}_1 K, \quad (9)$$

$$K = \tilde{Y}_1 R. \quad (10)$$

The third line follows from the second because $\tilde{L} \tilde{X}_1 = \text{Id}_r$. In particular K does not depend on the choice of $\tilde{L}$ or of R : it is the unique matrix with (8), since Y_1 has a right inverse.

Now (1) and (4) give $Y_1 Y_1^\dagger = X_2^\dagger B B^\dagger X_2 = \text{Id}_\ell$, because B is a unitary dressing. Hence $Y_1^\dagger$ is a right inverse of Y_1 and the second printed expression in (5) is exactly (10); it is correct as printed.

The first printed expression is not. By (2) the matrix $\tilde{X}_1^\dagger$ is a left inverse of $\tilde{X}_1$ only after the weight is inserted, so the available left inverse is $\tilde{L} = \tilde{X}_1^\dagger W_\rho$ and the comparison matrix is

$$K = \tilde{X}_1^\dagger W_\rho X_1. \quad (11)$$

Deleting W_ρ from (11) reproduces the printed formula, and the two agree exactly when $\tilde{X}_1^\dagger (W_\rho - \text{Id}_{dD}) X_1 = 0$, hence in particular when $\rho = \text{Id}_D$. Since ρ is the normalized transfer fixed point, that equality is not available in general. The printed $\tilde{X}_1^\dagger$ is therefore the same shorthand for the weighted adjoint that Section III of Ref. [CPGSV17] uses throughout: every occurrence of the adjoint of $\tilde{X}_1$ in the role of a left inverse is an occurrence of $\tilde{X}_1^\dagger W_\rho$. Reading it that way makes the appendix comparison correct with no further change, which is why the weight is a local correction rather than a defect in the argument.

The formal development uses (11) as the definition and proves (10) as a theorem.³

²Source traceability: `Papers/2502.20257/main.tex`, lines 5432–5443, in the proof of the second appendix proposition of the $\mathbb{Z}_2$ appendix.

³Formal declarations: `MPOTensor.inverseCompatibleComparisonK` and `MPOTensor.inverseCompatibleComparison` in `TNLean/MPS/MPU/InverseCompatibleCutComparison`; the weighted normalization is `MPOTensor.sourceX_weighted_isometry`.

3 The inverse comparison

Equation (7) asserts more than invertibility of K : it names $K^\dagger$ as the inverse, which is the statement that K is unitary. In the appendix that unitarity is obtained afterwards, from the pleasant properties reestablished for the new factorization.

The same construction applied in the opposite direction supplies an inverse without any normalization argument. If L is a left inverse of X_1 and $\tilde{Z}_1$ a right inverse of $\tilde{Y}_1$, put $J = L\tilde{X}_1$. Then, by (8)–(10) with the two factorizations exchanged,

$$Y_1 = J\tilde{Y}_1, \quad (12)$$

$$\tilde{X}_1 = X_1 J, \quad (13)$$

$$J = Y_1 \tilde{Z}_1, \quad (14)$$

and substituting each pairing into the other gives

$$KJ = \text{Id}_r, \quad (15)$$

$$JK = \text{Id}_\ell. \quad (16)$$

Thus J occupies the position of $K^\dagger$ in (7), and $J = K^\dagger$ holds precisely when K is unitary. The formal statement keeps J as constructed and asserts (12)–(16) as constructed. For a simple canonical-form-II tensor whose physical adjoint is carried by a unitary gauge with $T\bar{T} = \sigma I_D$, both K unitary and $J = K^\dagger$ are now formalized, so the pairing holds in its printed $K^\dagger$ form. The formal statement still keeps the rectangular index types r and ℓ rather than transporting along the proved equality $r = \ell$.⁴

4 Verdict

The second printed expression for the comparison matrix is correct. The first is the usual shorthand in which the adjoint of $\tilde{X}_1$ stands for the weighted adjoint $\tilde{X}_1^\dagger W_\rho$, and it is correct only under that reading; the formalization inserts the weight. This is a local correction, it changes no conclusion of the appendix argument, and it is carried out in full, so it leaves nothing open.

The identification of the inverse comparison with $K^\dagger$ is a separate statement that the appendix obtains from the pleasant properties of the new factorization. It was the live content of this note, and the reason for the scope-restriction classification: the formal development at first proved only that K has a two-sided inverse, exhibited explicitly, so its pairing was the printed pairing restricted to that explicit inverse rather than $K^\dagger$.

⁴Formal declarations: `MPOTensor.inverseCompatibleComparisonJ` and `MPOTensor.rightRank_eq_leftRank_of_physicalAdjointTensor_eq_unitary_gauge` in `TNLean/MPS/MPU/InverseCompatibleCutComparison`; `MPOTensor.inverseCompatibleComparisonK_isUnitaryBetween` and `MPOTensor.inverseCompatibleComparisonJ_eq_conjTranspose` in `TNLean/MPS/MPU/InverseCompatibleComparisonUnitarity`.

That restriction is now eliminated. For a simple canonical-form-II tensor whose physical adjoint is carried by a unitary gauge with $T\bar{T} = \sigma I_D$, the weighted comparison is unitary, and substituting that unitarity into (15) gives $J = K^\dagger$. The pairing therefore holds in the printed form, on the hypotheses the appendix itself uses, and the note is resolved. What remains is presentational only: the formal statement keeps the rectangular index types r and ℓ instead of transporting along the proved equality $r = \ell$.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.