

Source Boundary for the MPU Gauss Laws and Their Commuting Projectors

2026-08-29

Verdict: open-gap (open).

1 Scope

Proposition 7 of Ref. [FRBC25, lines 3378–3499] defines a local unitary representation $\mathcal{G}$ and the corresponding Gauss-law projectors P_j . Proposition 8 then states that the projectors commute if and only if the MPU symmetry is nonanomalous, and refers to Section 3.2 of Ref. [Sei24] for the adjacent-projector calculation [FRBC25, lines 3602–3655].

This note fixes the mathematical meaning of that citation before either proposition is formalized. It records the oriented local Hilbert space, the multiplication convention, the hypotheses used by the three-defect calculation, and the exact boundary of the existing formal library. It does not replace the cited tensor calculation by an abstract cocycle argument. The Seifnashri line ranges below refer to *Lieb-Schultz-Mattis anomalies as obstructions to gauging (non-on-site) symmetries*, arXiv:2308.05151v3, source file `LSMcocycle.tex`. This version is distributed under the arXiv non-exclusive distribution license 1.0. The formalization prerequisite is tracked by GitHub issue #7339.

2 Oriented local Hilbert space

Let G be a finite group and let the matter space at site j be $\mathcal{H}_j \cong \mathcal{H}$. In the Hamiltonian construction of Ref. [Sei24, Section 3.2], the chain is periodic, a defect g_j lies on the oriented link $(j, j+1)$, and the gauge basis on that link is written $|g_j\rangle$. The local transformation at j uses two matter factors and the two links adjacent to j . After the harmless reordering of tensor factors used in Ref. [FRBC25, lines 3380–3386], its space is

$$\mathcal{K}_j = \mathcal{H}_j \otimes \mathcal{H}_{j+1} \otimes \mathbb{C}[G]_{(j-1,j)} \otimes \mathbb{C}[G]_{(j,j+1)}. \quad (1)$$

The first gauge label is therefore incoming and the second is outgoing. If their values are a and b , a transformation by g sends

$$(a, b) \mapsto (ag^{-1}, gb). \quad (2)$$

Thus the incoming link carries right multiplication by g^{-1} , while the outgoing link carries left multiplication by g . Reversing these roles, or replacing either action by multiplication on the other side, changes the representation law.

Seifnashri first uses defect Hilbert spaces which may depend on all link labels. For a discrete group, the formula used here is obtained after the explicit simplifying assumption that every defect Hilbert space is an identical copy of the untwisted matter space. The extended space then becomes

$$\mathcal{H}_{\text{ext}} = \left(\bigotimes_{j=1}^L \mathcal{H}_j \right) \otimes \left(\bigotimes_{j=1}^L \mathbb{C}[G]_{(j,j+1)} \right), \quad (3)$$

with one matter factor per site and one gauge factor per oriented link [Sei24, Section 3.2, arXiv source lines 1637–1678].

3 The local representation and its multiplication convention

For $a, b, c, d \in G$ with $ab = cd$, let $\lambda_{a,b}^{c,d}$ be the unitary comparison of the two defect-pair factorizations. The MPU paper defines it from right fusion operators by $\lambda_{a,b}^{c,d} = (\lambda_{c,d}^R)^\dagger \lambda_{a,b}^R$ and proves the composition law

$$\lambda_{a,b}^{r,s} = \lambda_{c,d}^{r,s} \lambda_{a,b}^{c,d}, \quad ab = cd = rs. \quad (4)$$

These are the comparison operators of Ref. [FRBC25, lines 2782–2821]; they are not new fusion tensors.

On the oriented space in Eq. 1, the local operator is

$$\mathcal{G}_g^j = \sum_{a,b \in G} \lambda_{a,b}^{ag^{-1},gb} \otimes |ag^{-1}, gb\rangle\langle a, b|. \quad (5)$$

The superscript j records the support and is suppressed in the displayed formula (109) of Ref. [FRBC25, lines 3380–3386]. Applying first h and then g to one gauge sector gives

$$(a, b) \xrightarrow{\mathcal{G}_h^j} (ah^{-1}, hb) \xrightarrow{\mathcal{G}_g^j} (ah^{-1}g^{-1}, ghb) = (a(gh)^{-1}, (gh)b), \quad (6)$$

$$\lambda_{ah^{-1}, hb}^{ah^{-1}g^{-1}, ghb} \lambda_{a,b}^{ah^{-1}, hb} = \lambda_{a,b}^{a(gh)^{-1}, (gh)b}. \quad (7)$$

Equation 7 is precisely Eq. 4. Hence the convention is

$$\mathcal{G}_g^j \mathcal{G}_h^j = \mathcal{G}_{gh}^j, \quad (\mathcal{G}_g^j)^\dagger = \mathcal{G}_{g^{-1}}^j. \quad (8)$$

This agrees with the explicit convention in Ref. [Sei24, arXiv source lines 1691–1734]. Unitarity is not a consequence of the label permutation alone: every nonzero block $\lambda_{a,b}^{ag^{-1},gb}$ must be unitary. In Seifnashri’s construction this

comes from unitary defect fusion operators $\lambda^j(g, h)$; the link-extended operators $\tilde{\lambda}^j(g, h)$ are partial operators and are explicitly not unitary [Sei24, arXiv source lines 709–744 and 1680–1729].

The Gauss-law projector is the group average

$$P_j = \frac{1}{|G|} \sum_{g \in G} \mathcal{G}_g^j. \quad (9)$$

Equation 8 makes $P_j^2 = P_j = P_j^\dagger$. This is the local projection onto the invariant subspace; the additional constraint is $P_j = 1$ on physical states.

4 The three-defect associator calculation

The commuting-projector proof uses more than the scalar 3-cocycle equation. Let $\omega: G^3 \rightarrow \mathbb{C}^\times$ be the anomaly 3-cocycle defined by the MPU fusion tensors in Ref. [FRBC25, lines 1506–1545]. Let $T_{g(hk)}$ and $T_{(gh)k}$ denote the two unitary fusion paths for three consecutive defects in Equation (118) of Ref. [FRBC25, lines 3602–3655]. In the notation of its Appendix C [FRBC25, lines 6040–6961], they satisfy

$$T_{g(hk)} = F(g, h, k) T_{(gh)k}. \quad (10)$$

The left path contains the movement operator $w_R^g = \lambda_{g,e}^R$ between the fusions of (h, k) and (g, hk) ; the right path fuses (g, h) and then (gh, k) . The appendix computes the two tensor networks and obtains

$$F(g, h, k) = \omega(g, h, k). \quad (11)$$

This is a tensor-level result, proved at lines 6040–6962 of Ref. [FRBC25]; it must not be inferred merely from the existence of an independently supplied scalar cochain.

To compare $P_j P_{j-1}$ with $P_{j-1} P_j$, the cited calculation expands both products in defect sectors. The bottom labels g_1, g_2, g_3 and top labels h_1, h_2, h_3 have equal total product. A unitary fusion of the three top defects converts equality of the two expanded operators into equality of the two three-defect networks. Unitarity then cancels every fusion bubble through $\lambda^j(g, h)(\lambda^j(g, h))^\dagger = 1$. Applying an F -move on each side, followed by two further F -moves on the right, gives exactly

$$\frac{F(h_1, h_2 h_3 g_3^{-1}, g_3)}{F(g_1, g_2, g_3)} = \frac{F(h_1, h_2, h_3)}{F(g_1, g_1^{-1} h_1 h_2, h_3)}. \quad (12)$$

This is the calculation at arXiv source lines 1745–1955 of Ref. [Sei24], relabelled as in lines 3607–3650 of Ref. [FRBC25]. The denominators are legitimate because the fusion paths are unitary, so F has unit modulus in the chosen gauge.

If the projectors commute, set $h_1 = e$, $h_2 = g_1$, and $h_3 = g_2g_3$ in Eq. 12. One obtains

$$F(g_1, g_2, g_3) = \frac{F(g_1, e, g_2g_3)F(e, g_1g_2, g_3)}{F(e, g_1, g_2g_3)}. \quad (13)$$

The standing identity gauges give $F(e, g, h) = F(g, e, h) = 1$, hence $F(g_1, g_2, g_3) = 1$. Conversely, pointwise $F = 1$ makes Eq. 12 automatic. This is the actual “if and only if” calculation. The scalar cocycle identity does not by itself imply either direction.

5 Gauge-class triviality versus a pointwise trivial representative

The MPU paper calls a representation nonanomalous when the cohomology class of ω is trivial. This is an assertion about the orbit under fusion gauges, not the pointwise equation $\omega(g, h, k) = 1$ for an arbitrary representative. Proposition 6 of Ref. [FRBC25] first constructs a compatible fusion/action gauge in which all fusion operators are unitary and, in the nonanomalous case, the selected representative satisfies

$$\omega(g, h, k) = 1 \quad \text{for every } g, h, k \in G. \quad (14)$$

Only after this choice may Eq. 11 and Eq. 13 identify nonanomalous with commuting projectors. Seifnashri makes the same distinction: the direct calculation says $F^j = 1$, while the obstruction is the phase-redefinition class $[F^j]$ [Sei24, arXiv source lines 1541–1619 and 1956–1960].

The existing scalar API distinguishes class-triviality from the three identity-axis normalization conditions, but only algebraically over $\mathbb{C}^\times$. Both scalar 2-cochains and scalar 3-cochains take values in $\mathbb{C}^\times$ (the type `Units C`); `TNLean.Algebra.ScalarThreeCochain.IsTrivialGaugeClass` expresses class-triviality, while `TNLean.Algebra.ScalarThreeCochain.IsNormalized` expresses the three identity-axis normalizations. No present predicate restricts these units to the unit-modulus subgroup $U(1)$. Consequently, this API does not yet encode faithfully the phase-valued fusion gauge required by Proposition 6 of Ref. [FRBC25, lines 3268–3346]; that phase/unitarity restriction remains an explicit formal boundary. The API also does not package an H^3 quotient or attach a scalar cochain to fusion tensors.¹

6 Finite-chain support convention

Seifnashri’s Section 3.2 uses a periodic chain of L sites. Earlier, the defect width is defined independently of the system size, and the discussion then fixes width

¹Formal declarations are in `TNLean/Algebra/ScalarThreeCocycle`. The corrected three-axis normalization is documented in `docs/paper-gaps/fbc25_three_cocycle_normalization_typo`.

$k = 1$ so that adjacent defects and fusion operators are unambiguous [Sei24, arXiv source lines 555–584 and 602–760]. With this convention, $\mathcal{G}_g^j$ is supported on matter sites $j, j + 1$ and gauge links $(j - 1, j), (j, j + 1)$. Thus projectors whose supports are separated commute by tensor-factor locality, and the only load-bearing overlap is the adjacent pair P_j, P_{j-1} .

The adjacent calculation uses three distinct matter sites $j - 1, j, j + 1$ and three distinct gauge links from $(j - 2, j - 1)$ through $(j, j + 1)$. Thus $L \geq 3$ is the natural stable finite-periodic convention. For $L = 3$, the two endpoint names $j - 2$ and $j + 1$ coincide, but the three matter factors and three oriented link factors used by the operators remain distinct; every pair of different projectors is adjacent. For $L = 2$, the three-link sector expansion identifies gauge factors and is not the cited calculation. The papers do not separately verify this wrapped case. Consequently, a theorem restricted to $L \geq 3$ does not by itself formalize an unqualified all-length reading that includes $L = 2$; that case requires a separate calculation.

The current finite-region API supplies identity-tensor inclusions and commutation for disjoint subsets of $\mathbb{Z}$.² It does not yet represent the heterogeneous oriented factors in Eq. 1, nor cyclic wrap-around. Conversely, `MPOTensor.IsMPU` and `MPOTensor.IsMPU.mpo_mem_unitaryGroup` describe homogeneous periodic MPO operators for every $N > 1$, but not an arbitrary-boundary local operator with separate matter and link factors.³ Neither surface is by itself a definition of $\mathcal{G}_g^j$ or P_j .

7 Existing ownership boundaries

The remaining work should use the following existing declarations and issue boundaries.

1. The comparison operators $\lambda_{a,b}^{c,d}$ and Eq. 4 belong to GitHub issue #7334. That issue uses the paper’s definition $(\lambda_{c,d}^R)^\dagger \lambda_{a,b}^R$ and standard matrix-unitary multiplication.
2. Unitary λ^L, λ^R and the gauge in Eq. 14 belong to GitHub issue #7336. That issue owns the trace normalization and the compatible fusion/action gauge.
3. The fusion operators and anomaly data needed for the associator comparison belong to GitHub issue #7322. The tensor theorem $F = \omega$ in Eq. 11 belongs to GitHub issue #7359, which depends on GitHub issue #7322. The scalar must be derived from the two MPU fusion trees.

²Formal declarations: `SpinChain.LocalAlgebra`, `SpinChain.localInclusion`, and `SpinChain.localInclusion_commute_of_disjoint` in `TNLean/QCA/LocalAlgebra` and `TNLean/QCA/D` `isjointSupport`.

³Formal declarations are in `TNLean/MPS/MPU/Basic`.

4. The identity and inverse gauges used in Eq. 13 belong to GitHub issue #7325. Their normalized identity cases should be reused rather than postulated again.
5. Scalar class and normalization are already expressed by `TNLean.Algebra.ScalarThreeCochain.CohomologousTo`, `TNLean.Algebra.ScalarThreeCochain.IsTrivialGaugeClass`, and `TNLean.Algebra.ScalarThreeCochain.IsNormalized`.
6. Disjoint finite support is already expressed by `SpinChain.localInclusion_commute_of_disjoint`. It closes only the separated-support part after a faithful oriented embedding has been given.

Matrix unitarity itself is already expressed by `Matrix.unitaryGroup`, with the two equations exposed by `Matrix.mem_unitaryGroup_iff` and `Matrix.mem_unitaryGroup_iff'`. QICLean supplies the matrix C^* -algebra infrastructure used by the finite-region API, but a repository search finds no defect-fusion, Gauss-law, or oriented matter-link representation that would make Eq. 5 redundant.

8 Conclusion

The citation fixes a left representation $\mathcal{G}_g^j \mathcal{G}_h^j = \mathcal{G}_{gh}^j$ on two matter factors and the incoming/outgoing gauge links, with the incoming label transformed by right multiplication by g^{-1} and the outgoing label by left multiplication by g . The commuting-projector result depends on the full adjacent three-defect tensor calculation, unitary bubble cancellation, the theorem $F = \omega$, the normalized identity gauges, and a subsequent choice of a pointwise trivial representative of a trivial anomaly class.

The open formal boundary is therefore not scalar cohomology. It is the construction and oriented finite-chain placement of the unitary comparison operators, followed by the exact calculation in Eq. 12. The stable-range support argument is $L \geq 3$; an unrestricted finite-chain theorem also owes the $L = 2$ wrapped case. No implementation of Proposition 8 of Ref. [FRBC25, lines 3602–3655] is claimed here.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.
- [Sei24] Sahand Seifnashri. Lieb-schultz-mattis anomalies as obstructions to gauging (non-on-site) symmetries. *SciPost Phys.*, 16:098, 2024.