

Compatibility of CZX Fusion and Action Coordinates

2026-09-06

Verdict: open-gap (wip).

1 The comparison to be established

The definition of the fusion–action scalar in (62) of Ref. [FRBC25, lines 1875–1913] compares a right fusion cap with two successive right action tensors. The neutral action is trivial, with bond dimension one [FRBC25, line 1937]. Write q_x for the printed right action column on the GHZ block x , a vector in $\mathbb{C}^2$, and p_x for the printed right action row. For the CZX generator g , which exchanges the two GHZ blocks, the right boundary on the successive-action side is therefore $q_{1-x} \otimes q_x$, where $x \in \{0, 1\}$ is the initial block. The source later gives the scalars $L_{g,g}^0 = -1$ and $L_{g,g}^1 = 1$, as well as physical fusion identities with these signs [FRBC25, lines 4695–4699 and 4989–5067].

The maintained cap/action coordinates specified here give successive-action proportionality coefficients $c_0 = 1$ and $c_1 = -1$, where c_x is the scalar by which the fusion cap and the successive action columns differ after one supported product letter on block x . This is a nonzero, positive-length obstruction to identifying the two calculations in those coordinates. It is not yet a proof that the source assertion is false: that conclusion requires a common interpretation of every oriented leg, density insertion, and fusion gauge. In particular, a finite contraction with well-defined indices need not be the contraction intended by an incompletely typed diagram.

2 Fixed coordinates and oriented boundaries

Write a physical index as $j = 2a + b$, with $a, b \in \{0, 1\}$, and use the computational basis in this order. The GHZ block x has coordinates $A_x^j = \delta_{j, 3x}$. All matrices below have output indices on rows and input indices on columns; boundary covectors are rows and boundary vectors are columns. No additional complex conjugation is applied when a row is contracted with a column. In the product virtual space the ordered pair (α, γ) has index $2\alpha + \gamma$, with the upper symmetry tensor first. The interior-action diagram places the left action

tensor at the left boundary and the right action tensor at the right boundary [FRBC25, lines 1841–1855]. The right fusion symbol has its two unfused legs to the left, ordered upper then lower; the right action symbol likewise has the symmetry leg above the block leg.¹

The blocked CZX tensor of (160) in Ref. [FRBC25, lines 4505–4547], including its two output Z gates and unnormalized Hadamard weights, has coordinates

$$T_{\alpha\beta}^{2a+b, 2c+d} = \delta_{a,1-c} \delta_{b,1-d} \delta_{\alpha,a} (-1)^{a+b+ab+b\beta}. \quad (1)$$

Here α is the left virtual index and β the right virtual index. The printed action tensors and the right fusion cap in Ref. [FRBC25, lines 4671–4694] are read as

$$\begin{aligned} p_0 &= (0, -i), & p_1 &= (1, 0), \\ q_0 &= \tfrac{1}{2}(-i, i)^\top, & q_1 &= \tfrac{1}{2}(1, 1)^\top, \\ W_{2\beta+\gamma} &= \tfrac{1}{2}Y_{\gamma\beta}, & Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & W &= \tfrac{1}{2}(0, i, -i, 0)^\top. \end{aligned} \quad (2)$$

The action columns use $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. In particular, the ordered cap in (2) uses $Y_{\gamma\beta}$, not $Y_{\beta\gamma}$.

Let B_x be the supported letter of the product of the two symmetry tensors on block x . Its left and right product indices are (α, γ) and (β, δ) , respectively:

$$(B_x)_{2\alpha+\gamma, 2\beta+\delta} = \sum_{j=0}^3 T_{\alpha\beta}^{3x,j} T_{\gamma\delta}^{j,3x}, \quad (3)$$

$$P_x = p_{1-x} \otimes p_x, \quad Q_x = q_{1-x} \otimes q_x. \quad (4)$$

The scalar neutral action contributes the number 1 on the fused side. Thus the candidate coordinate reading of the defining comparison is $B_x W = c_x B_x Q_x$, with the same order of virtual legs on both sides.

Direct substitution of (1) into (3) and (4) gives

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5)$$

$$\begin{aligned} P_0 &= (0, -i, 0, 0), & P_1 &= (0, 0, -i, 0), \\ Q_0 &= \tfrac{1}{4}(-i, i, -i, i)^\top, & Q_1 &= \tfrac{1}{4}(-i, -i, i, i)^\top. \end{aligned} \quad (6)$$

These are ordinary finite matrices; their definition does not presume an identification with the source’s L -symbols.

¹The direct orientation anchors are `Papers/2502.20257/main.tex`, lines 1841–1855, and the drawing definitions in `Papers/2502.20257/Commands.tex`, lines 324–338 for right fusion and lines 386–399 for right action.

3 The positive-length obstruction

Let e_0, e_1, e_2, e_3 denote the standard basis of the product virtual space. Multiplication of the matrices in (5)–(6) gives

$$\begin{aligned} B_0 Q_0 &= i e_1, & B_0 W &= i e_1, \\ B_1 Q_1 &= i e_2, & B_1 W &= -i e_2, \\ P_x B_x Q_x &= 1, & B_x^2 &= B_x. \end{aligned} \quad (7)$$

Consequently the coefficient is uniquely determined:

$$\begin{aligned} c_x &:= P_x B_x W, \\ c_0 &= P_0(i e_1) = (-i)(i) = 1, \\ c_1 &= P_1(-i e_2) = (-i)(-i) = -1, \\ B_x W &= c_x B_x Q_x. \end{aligned} \quad (8)$$

The two substitutions pair the values of $B_0 W$ and $B_1 W$ in (7) with the second and third entries of P_0 and P_1 in (6). Indeed, applying P_x to any proposed proportionality and using (7) recovers its coefficient. For every integer $n \geq 1$, idempotence also gives

$$B_x^n = B_x, \quad P_x B_x^n Q_x = 1, \quad B_x^n W = c_x B_x^n Q_x. \quad (9)$$

By (9) the discrepancy therefore persists at every positive repetition of the supported letter. It does not arise from an empty product. The separate empty-word correction to the action pair neither explains nor removes (8).²

4 The retained physical operator

For two blocked sites, number the four qubits 0, 1, 2, 3 from left to right. Read circuits from bottom to top and operator products from right to left. The unmodified physical operator printed in Ref. [FRBC25, lines 4989–5013] is

$$\lambda = -i(X_0 X_1 X_2 Z_3) CZ_{01} CZ_{12} X_2. \quad (10)$$

The displayed defect evaluations are $g[0] = |00\rangle$ and $g[1] = -i|11\rangle$ [FRBC25, lines 4800–4859]. Retain these evaluations and set $e[x] = |xx\rangle$. With the left defect first, define

$$v_x = g[1 - x] \otimes g[x], \quad u_x = e[x] \otimes e[x], \quad (11)$$

$$v_0 = -i|1100\rangle, \quad v_1 = -i|0011\rangle, \quad u_0 = |0000\rangle, \quad u_1 = |1111\rangle. \quad (12)$$

²See `docs/paper-gaps/fbc25_czx_action_empty_word.tex`. The coordinate statements are recorded in `MPOTensor.CZX.fusionActionCoefficient_coordinates`, `MPOTensor.CZX.fusionAction_finite_identities`, and `MPOTensor.CZX.fusionAction_positive_power`, in `TNLean/MPS/MPD0/CZXFusionActionCoordinates`. These names identify the explicit contractions, not a theorem equating them with the source's L -symbols.

The retained operator of (10) applied to the vectors of (12) gives precisely the printed signs [FRBC25, lines 5014–5067]:

$$\lambda v_0 = -u_0, \quad \lambda v_1 = u_1. \quad (13)$$

Thus the physical coefficients in (13) are $-c_x$, not c_x .³ This comparison does not alter λ , the cap, the block labels, or the displayed defects. In particular, it does not replace the unmodified operator by the modified fusion operator.

5 Two orientation diagnostics

The displayed factorization in Ref. [FRBC25, lines 4559–4659] admits the following explicit index convention. For $j = 2a + b$, $k = 2c + d$, and $\bar{j} = 3 - j$, define factors with the indicated row and column indices by

$$(X_2)_{(\alpha,j),k} = \delta_{j,k} \delta_{\alpha,a} (-1)^{a+b}, \quad (14)$$

$$(Y_2)_{k,(j,\beta)} = \delta_{k,\bar{j}} (-1)^{cd+d\beta},$$

$$(X_1)_{(j,\beta),k} = \sum_{t=0}^1 Y_{\beta t} \overline{(Y_2)_{k,(j,t)}},$$

$$(Y_1)_{k,(\alpha,j)} = \sum_{t=0}^1 \overline{(X_2)_{(t,j),k}} Y_{t\alpha}. \quad (15)$$

Bars on entries denote complex conjugation; $\bar{j}$ denotes only the complemented physical index. These formulas specify the direction of every horizontal Y factor.

With identity on the one-dimensional fused endpoint, consider the fully typed two-site contraction

$$\Lambda_{(j,r),(k,l)} = \sum_{\alpha,\beta,\gamma=0}^1 \sum_{s=0}^3 (X_1)_{(j,\alpha),k} T_{\alpha\beta}^{r,s} (X_1)_{(s,\gamma),l} W_{2\beta+\gamma}. \quad (16)$$

The free output indices are (j, r) and the free input indices are (k, l) , each in left-site, right-site order. A separate coordinate diagnostic yields

$$\Lambda_{(j,r),(k,l)} = -\langle j, r | \lambda | k, l \rangle. \quad (17)$$

This is not an identification of (16) with the general source definition. In (91b) of Ref. [FRBC25, lines 2675–2704], the density insertion is labelled ρ_g on the fused bond. For $g = h$ in this example that bond has dimension $D_{gh} = D_e = 1$, whereas $D_g = 2$. Interpreting that insertion requires an explicit domain and codomain; silently replacing it by the identity on the fused endpoint is not a source justification.

³The two physical contractions are `MPOTensor.CZX.matterMatrix_lambda_mulVec_defectVector_zero` and `MPOTensor.CZX.matterMatrix_lambda_mulVec_defectVector_one`, in `TNLean/MPS/MPDO/CZXUnmodifiedFusion`. They concern the unmodified physical operator and do not identify its coefficients with a virtual fusion–action scalar.

There is a second diagnostic at one site:

$$\sum_{\alpha=0}^1 p_0(\alpha)(Y_1)_{0,(\alpha,0)} = -1, \quad \sum_{\alpha=0}^1 p_1(\alpha)(Y_1)_{3,(\alpha,3)} = i. \quad (18)$$

These are opposite to the displayed defect coefficients. The convention used to recover the displayed defects takes the outgoing bond amplitude on block x to be $Y_{\alpha x}$; the factor in (15) supplies $Y_{x\alpha}$. Since $Y^\top = -Y$, these readings differ by a sign. The existing defect-domain note fixes its reading by the displayed defect evaluations; it does not settle the compatibility of that reading with all four factors in (14)–(15).⁴

6 Verdict and elimination criterion

The classification is an open gap under active investigation. The explicit boundary coefficients, their uniqueness, and their persistence at positive length are determined. The retained physical signs are also determined. What remains open is whether a single source-faithful convention identifies these two constructions. The present result is not a verdict that the paper contains a proved error, nor does it invalidate the finite restrictions of the retained physical circuits on the displayed defect vectors.

To eliminate the gap, specify the domain and codomain of every factor in the fusion and defect diagrams, including the fused density insertion, and write all contractions with common physical and virtual index orders. The same convention must recover the blocked tensor, both displayed decompositions, the printed fusion and action tensors, the neutral action, the displayed defects, and the unmodified physical fusion circuit. It must then derive the fusion–action comparison and the physical defect identities within that convention. If a representation or gauge transformation intervenes, its effect on every one of these objects and on the scalar must be exhibited, rather than changing only the sign at the disputed step.

No cap negation, scalar gauge flip, block relabelling, dressed action tensor, or change of the modified fusion operator is adopted here to force agreement. If a common convention is found, the note can be resolved by giving the explicit identifications. If the source fixes incompatible conventions and no such identification is possible, the precise affected assertion and its correction must first be isolated. The coordinate obstruction alone does not settle broader completion or gauging questions.

⁴See `docs/paper-gaps/fbc25_czx_defect_domains.tex`. The calculations (17) and (18) remain diagnostic calculations: neither carries a formal declaration, and no identification of either with the intended source diagram is asserted here.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.