

CZX Defect Domains from the Source Diagrams

2026-09-04

Verdict: clarification (resolved).

1 The assertion

The state-level gauging of the anomalous $\mathbb{Z}_2$ CZX symmetry in Ref. [FRBC25, lines 4503–5183] rests on the modified-fusion lemma of Ref. [FRBC25, lines 4215–4254]. That lemma attaches to every ordered pair of group labels (g, h) and every injective block x a two-site defect vector $v(g, h, x)$, drawn as the contraction of the defect tensor $g[hx]$ on the left site with the defect tensor $h[x]$ on the right site, and prescribes unitary fusion operators $\tilde{\lambda}_{g,h}$ on the two-site matter space with

$$\tilde{\lambda}_{g,h} v(g, h, x) = v(e, gh, x). \quad (1)$$

For the CZX example the source displays the blocked symmetry tensor, its two decompositions, the two injective blocks of the invariant GHZ state, the action tensors, the two domain-wall defect tensors, the movement operator, the fusion operators, and the modification of the (g, g) fusion operator. It does not display the eight two-site vectors $v(g, h, x)$ themselves, nor the four defect domains

$$\mathcal{K}_{g,h} = \text{span}\{v(g, h, x) : x \in \{0, 1\}\} \quad (2)$$

on which Eq. 1 constrains the fusion operators. Unpublished planning material for the gauge-invariant-subspace problem therefore treated two statements as explicit axioms: that the retained four-qubit circuits coincide with the source’s fusion operators in one common blocking, basis, phase, and fusion gauge, and that the odd target support $\text{span}\{v(e, g, x) : x\}$ coincides with the even one $\text{span}\{v(e, e, x) : x\}$. This note derives all eight vectors by finite contraction of the displayed tensors and settles both statements. The first is a theorem. The second is false: the two target supports are orthogonal.

2 Conventions read off the diagrams

One blocked site carries two qubits. Two neighboring blocked sites carry the qubits 0, 1 (left site) and 2, 3 (right site), and the computational basis of the

sixteen-dimensional two-site matter space is written $|x_0x_1x_2x_3\rangle$. The two injective blocks of the once-blocked GHZ state are the product vectors

$$|\circ\rangle = |00\rangle, \quad |\bullet\rangle = |11\rangle, \quad (3)$$

drawn as a white and a black circle [FRBC25, lines 4660–4670]. Both blocks have bond dimension one, so every virtual contraction between neighboring defect tensors is a scalar and every single-site defect tensor is a two-qubit vector.

Circuit diagrams are read from the bottom (input) to the top (output), and operator products act from right to left. The controlled- Z gate is drawn as a bond of dimension two between the two qubit lines with a blue dot on the bond [FRBC25, lines 3800–3845]: the attachment on a qubit line is the copy tensor $\delta_{i,j,k}$, which exports the qubit value onto the bond, and the blue dot is the matrix

$$\sqrt{2}H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{CZ}_{rs}|x\rangle = (-1)^{x_r x_s}|x\rangle. \quad (4)$$

The blocked symmetry tensor (160) of Ref. [FRBC25, lines 4505–4545] applies $X \otimes X$ at the bottom, then a controlled- Z between its two qubits, exports the first qubit onto the left bond and the second qubit onto the right bond (with the blue dot on the right bond), and applies $Z \otimes Z$ at the top. The source picks two decompositions of this tensor [FRBC25, lines 4559–4659]. The one used in the definition of the defect tensors is the white tensor with a wavy output leg and a bond leg to the left: it applies $Z \otimes Z$, exports the first qubit onto the bond through a copy tensor, and carries the Pauli matrix Y on the bond. On a block vector $|xx\rangle$ its output is therefore

$$|xx\rangle \otimes Y|x\rangle = i(-1)^x |xx\rangle \otimes |x+1\rangle, \quad (5)$$

where the second factor lives on the bond. The truncated symmetry displayed in Ref. [FRBC25, lines 4700–4798], whose rightmost site carries this white tensor, has the overall prefactor i that Eq. 5 produces; this confirms the direction in which Y acts on the bond. The left action tensors are the bond covectors

$$A_{g,0}^\Gamma = -i\langle 1|, \quad A_{g,1}^\Gamma = \langle 0| \quad (6)$$

of Ref. [FRBC25, lines 4690–4694], labelled by the block x to the right of the defect.

3 The single-site defect vectors

The domain-wall defect $g[x]$ is defined in Ref. [FRBC25, lines 2831–2845] as the block tensor x at the bottom, the white tensor of Eq. 5 on its physical leg, and the left action tensor contracted against the bond. Since g exchanges the two blocks, the defect $g[1]$ carries block 0 to its left and block 1 to its right, which

the source writes $g[1] \equiv 0|1$, and similarly $g[0] \equiv 1|0$. Contracting Eq. 5 with Eq. 6 gives

$$g[1] = A_{g,1}^\Gamma(i(-1)^1|0\rangle)|11\rangle = -i|11\rangle, \quad (7)$$

$$g[0] = A_{g,0}^\Gamma(i|1\rangle)|00\rangle = |00\rangle. \quad (8)$$

These are exactly the evaluations displayed in Ref. [FRBC25, lines 4800–4859]: $g[1]$ equals $-i$ times the black circle and $g[0]$ equals the white circle. The displayed values fix the reading of the diagram. The transposed action of Y on the bond would give $g[1] = i|11\rangle$ and $g[0] = -|00\rangle$, contradicting both displayed evaluations. The alternative expressions displayed in the same lines, through the black tensor of the first decomposition and the right action tensors $-\frac{i}{\sqrt{2}}|-\rangle$ for block 0 and $\frac{1}{\sqrt{2}}|+\rangle$ for block 1, labelled by the block to the left of the defect, reproduce Eqs. 7 and 8 with the displayed signs, that is, with the sign function $\xi_g(1) = -1$ and $\xi_g(0) = 1$ of Ref. [FRBC25, lines 2925–2962].

The identity defect $e[x]$ is the block tensor itself,

$$e[0] = |00\rangle, \quad e[1] = |11\rangle. \quad (9)$$

This is the source’s convention: projecting the gauge degrees of freedom of the gauged tensor onto the neutral element recovers the original tensor [FRBC25, lines 3494–3529], and every displayed fusion identity in Ref. [FRBC25, lines 4889–5066] draws a fused identity defect as a plain block circle.

4 The two-site defect vectors, domains, and prescribed maps

The two-site vector $v(a, b, x)$ is the tensor product of the left defect $a[bx]$ with the right defect $b[x]$, in this order [FRBC25, lines 4234–4245]. Writing $e \leftrightarrow 0$ and $g \leftrightarrow 1$ for the labels, so that $bx = x + b$ in $\mathbb{F}_2$, Eqs. 7–9 give

$$v(e, e, 0) = |0000\rangle, \quad v(e, e, 1) = |1111\rangle, \quad (10)$$

$$v(e, g, 0) = |1100\rangle, \quad v(e, g, 1) = -i|0011\rangle, \quad (11)$$

$$v(g, e, 0) = |0000\rangle, \quad v(g, e, 1) = -i|1111\rangle, \quad (12)$$

$$v(g, g, 0) = -i|1100\rangle, \quad v(g, g, 1) = -i|0011\rangle. \quad (13)$$

Each pair $v(a, b, 0)$, $v(a, b, 1)$ is orthogonal, as the modified-fusion lemma requires. The four defect domains of Eq. 2 are

$$\mathcal{K}_{e,e} = \mathcal{K}_{g,e} = \mathcal{L}_0 := \text{span}\{|0000\rangle, |1111\rangle\}, \quad (14)$$

$$\mathcal{K}_{e,g} = \mathcal{K}_{g,g} = \mathcal{L}_1 := \text{span}\{|1100\rangle, |0011\rangle\}. \quad (15)$$

Here $\mathcal{L}_p = \text{span}\{v(e, p, x) : x\}$ is the target support of the product label p . The two supports are orthogonal,

$$\mathcal{L}_0 \perp \mathcal{L}_1, \quad \mathcal{L}_1 \neq \mathcal{L}_0. \quad (16)$$

The domain depends only on the right label b , because the single-site defects $g[x]$ and $e[x]$ are proportional.

The prescribed maps $D_{a,b}: \mathcal{K}_{a,b} \rightarrow \mathcal{L}_{a+b}$, $v(a, b, x) \mapsto v(e, a + b, x)$, of Eq. 1 are

$$D_{e,e} = \mathbb{1}_{\mathcal{L}_0}, \quad D_{e,g} = \mathbb{1}_{\mathcal{L}_1}, \quad (17)$$

$$D_{g,e}|0000\rangle = |1100\rangle, \quad D_{g,e}|1111\rangle = |0011\rangle, \quad (18)$$

$$D_{g,g}|1100\rangle = i|0000\rangle, \quad D_{g,g}|0011\rangle = i|1111\rangle. \quad (19)$$

Each is a unitary isomorphism between its domain and its target. The maps $D_{g,e}$ and $D_{g,g}$ exchange the two supports, while $D_{e,e}$ and $D_{e,g}$ preserve them.

5 The displayed fusion operators restrict to the prescribed maps

The movement operator w_R of Ref. [FRBC25, lines 4860–4888] acts on the four qubit lines with Z on qubits 2, 3 at the bottom, a controlled- Z between qubits 1, 2, a controlled- Z between qubits 0, 1, and X on qubits 0, 1 at the top. The fusion operator $\lambda_{g,g}^R$ of Ref. [FRBC25, lines 4989–5013] carries the prefactor $-i$, X on qubit 2 at the bottom, the same two controlled- Z gates, and X on qubits 0, 1, 2 together with Z on qubit 3 at the top. Read from bottom to top,

$$w = (X_0 X_1) \text{CZ}_{01} \text{CZ}_{12} (Z_2 Z_3), \quad (20)$$

$$\lambda = -i (X_0 X_1 X_2 Z_3) \text{CZ}_{01} \text{CZ}_{12} X_2, \quad (21)$$

$$\tilde{\lambda} = \lambda Z_0, \quad (22)$$

where Eq. 22 is the modification $\tilde{\lambda}_{g,g}^R = \lambda_{g,g}^R Z_1$ of Ref. [FRBC25, lines 5179–5183]. The source names this factor “the Pauli Z operator on the first qubit” and numbers the qubits of a site from 1, whereas the present note numbers the four qubits of the two sites 0, 1, 2, 3. The source’s Z_1 and the Z_0 of Eq. 22 are therefore one operator, the Pauli Z on the first qubit of the left site, written in the two numbering conventions.¹ With $\bar{x} = (x_0 + 1, x_1 + 1, x_2, x_3)$, the phase tables are

$$w|x\rangle = (-1)^{x_0 x_1 + x_1 x_2 + x_2 + x_3} |\bar{x}\rangle, \quad (23)$$

$$\lambda|x\rangle = -i(-1)^{x_0 x_1 + x_1 x_2 + x_1 + x_3} |\bar{x}\rangle, \quad (24)$$

$$\tilde{\lambda}|x\rangle = -i(-1)^{x_0 x_1 + x_1 x_2 + x_0 + x_1 + x_3} |\bar{x}\rangle. \quad (25)$$

These transcriptions reproduce every fusion identity the source displays. The two movement identities of Ref. [FRBC25, lines 4889–4988] read, in the vectors of Eqs. 11 and 12,

$$w v(g, e, 0) = w|0000\rangle = |1100\rangle = v(e, g, 0), \quad (26)$$

$$w v(g, e, 1) = -i w|1111\rangle = -i|0011\rangle = v(e, g, 1), \quad (27)$$

¹These are the gate forms transcribed in `TNLean/MPS/MPD0/CZXGaussCircuitTuple`, whose qubit numbering and phase tables the present note verifies against the diagrams.

which is the statement $\lambda_{g,e}^R = w_R$ with trivial L -symbols (107) of Ref. [FRBC25, lines 3331–3340]. The two identities displayed for $\lambda_{g,g}^R$ in Ref. [FRBC25, lines 5014–5066] read

$$\lambda v(g, g, 0) = -i \lambda |1100\rangle = -i (-i) |0000\rangle = -v(e, e, 0), \quad (28)$$

$$\lambda v(g, g, 1) = -i \lambda |0011\rangle = -i (i) |1111\rangle = v(e, e, 1), \quad (29)$$

which are the printed L -symbols $L_{g,g}^0 = -1$ and $L_{g,g}^1 = 1$ [FRBC25, line 5067]. The alternative top-to-bottom reading of the circuits would give $w v(g, e, 0) = -v(e, g, 0)$ and $\lambda v(g, g, 1) = -v(e, e, 1)$, contradicting the displayed identities; the displayed identities therefore pin the reading. The modified operator then satisfies

$$\tilde{\lambda} v(g, g, 0) = \lambda Z_0(-i |1100\rangle) = i \lambda |1100\rangle = |0000\rangle = v(e, e, 0), \quad (30)$$

$$\tilde{\lambda} v(g, g, 1) = \lambda Z_0(-i |0011\rangle) = -i \lambda |0011\rangle = |1111\rangle = v(e, e, 1), \quad (31)$$

which is Eq. 1 exactly. The Z on qubit 1 would do the same, because both vectors of $\mathcal{L}_1$ have equal first two bits; a Z on qubit 2 or 3, or the product $Z_0 \lambda$, would satisfy the prescription only up to a global sign -1 . The source’s choice, Z on the first qubit applied before $\lambda_{g,g}^R$, is the one that satisfies Eq. 1 without a sign.

Together with the trivial cases $\lambda_{e,e}^R = \lambda_{e,g}^R = \mathbb{1}$, Eqs. 26–31 show that the four displayed operators $(\mathbb{1}, \mathbb{1}, w, \tilde{\lambda})$, indexed by $(a, b) \in \{e, g\}^2$, restrict on the domains of Eqs. 14 and 15 to the prescribed maps of Eqs. 17–19.

6 Verdicts

Convention matching is a theorem. The retained four-qubit circuits are the source’s fusion operators in the conventions of the preceding sections: qubits 0, 1 on the left blocked site and 2, 3 on the right, circuits read from bottom to top, the controlled- Z decomposition of Eq. 4, the displayed prefactor $-i$, and the displayed fusion gauge $\tilde{\lambda}_{g,g}^R = \lambda_{g,g}^R Z_1$. Every one of these conventions is either stated by the source or forced by a displayed identity, and the restrictions of the circuits to the derived domains are the prescribed maps. No axiom is needed.

The single-image identification is false. The odd target support is

$$\mathcal{L}_1 = \text{span}\{|1100\rangle, |0011\rangle\} = \mathcal{K}_{g,g}, \quad (32)$$

which is orthogonal to $\mathcal{L}_0 = \text{span}\{|0000\rangle, |1111\rangle\}$ by Eq. 16. The source draws enough to fix $\mathcal{L}_1$, so this is not a source gap. The identification $\mathcal{L}_1 = \mathcal{L}_0$ assumed in the planning material yields the domains $\mathcal{K}_{e,g} = \mathcal{L}_0$ and $\mathcal{K}_{g,e} = \mathcal{L}_1$, which are the wrong way round: by Eqs. 14 and 15, $\mathcal{K}_{e,g} = \mathcal{L}_1$ and $\mathcal{K}_{g,e} = \mathcal{L}_0$. The support-level classification by a Grassmannian of rank-two odd targets

collapses: the source fixes the odd target, and the four-sector support-and-restriction system is unique. The support projector of the sector (a, b) is the orthogonal projector onto $\mathcal{L}_b$. Transports between sectors with different product labels are the maps $D_{g,e}$ and $D_{g,g}$ themselves, so no additional identification of the two supports is required or available.

Membership in the full completion class. Because the displayed tuple $(\mathbb{1}, \mathbb{1}, w, \tilde{\lambda})$ agrees with the prescribed isometries on all four domains, it is a member of the full unitary completion class of the CZX defect maps, the class of unitary tuples that extend the four prescribed maps. The exact dimension 2^N of its gauge-invariant subspace on $N \geq 3$ blocked sites is therefore a lower bound on the maximum of that dimension over the full class. The value 2^N is not stated in the source, whose outlook [FRBC25, line 5198] leaves the growth of this subspace to future work; it is a theorem of the formal development about the displayed tuple alone, proved from the explicit gates.² This lower bound was previously conditional on supplying the three undisplayed domains and maps; the derivation above supplies them.

An index convention that does not matter. The gauged tensor displayed in Ref. [FRBC25, lines 5131–5178] carries, for gauge label 1, the matrix with entries 1 and $-i$ off the diagonal. With rows indexed by the block to the right of the defect and columns by the block to its left, its entries are the scalars of Eqs. 8 and 7; with the opposite reading the two scalars are exchanged. Either reading gives the same four domains and the same four prescribed maps, since the scalars enter both sides of Eq. 1 consistently.

7 Formal state

The gap is resolved. Everything this note asserts is formal: the single-site defects of Eqs. 7–9, the eight two-site vectors of Eqs. 10–13, the domains of Eqs. 14 and 15, and the prescribed maps of Eqs. 17–19, packaged as one family of prescribed defect maps satisfying Eq. 1 on their domains, together with the orthogonality of Eq. 16 and the two-dimensionality of each domain. The four circuits and their phase tables are formal, as is the finite computation of Eqs. 26–31: both displayed tuples are members of the completion class of those prescribed maps, and the lower bounds on the gauge-invariant dimension over that class carry no conditional clause.

The minimum over the class, the exact value of the supremum, and which completion attains it lie outside this note, which concerns the defect data read off the diagrams. They are questions about the completion class itself, tracked by GitHub issue #7568, and the source raises neither: what Ref. [FRBC25, lines 5198–5204] leaves open is whether the gauge-invariant subspace of the

²The theorem is `finrank.commonFixedSubmodule.placedGaussProjector.circuitTuple` in `TNLean/MPS/MPDO/CZXGaussInvariantSubspace`.

state-level gauging stays bounded or grows with the system size. The CZX defect and fusion data are tracked by GitHub issue #7355, the prescribed defect maps by GitHub issue #7739, the completion membership of the displayed tuples by GitHub issue #7740, and the completion-class instance by GitHub issue #7569.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.