

The Empty Interior Word in the Printed CZX Action Pair

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Verdict: false-source (resolved).

1 The assertion and its boundary

The interior action identity (60) in Ref. [FRBC25, lines 1841–1870] includes words of length $n \geq 0$. On a bond-one block it requires

$$p_x B_x^{\mathbf{a}} q_x = A_{1-x}^{\mathbf{a}} \quad (|\mathbf{a}| \geq 0), \quad (1)$$

where B_x is the tensor obtained by acting with the CZX symmetry on GHZ block $x \in \{0, 1\}$, A_{1-x} is the target block, and p_x, q_x are the left and right action tensors. Products over the empty word are identities, so (1) requires $p_x q_x = 1$. The source prints a specific action pair in Ref. [FRBC25, lines 4690–4694]. That pair satisfies (1) for every nonempty word but not for the empty word. This note isolates that boundary and gives a different pair which satisfies the identity at every length. It does not change the all-word requirement of a reduction.

2 The printed pair and the acted tensor

Use the physical alphabet $2a + b$, with $a, b \in \{0, 1\}$. The once-blocked GHZ sectors are $|00\rangle$ and $|11\rangle$, and the generator exchanges their labels [FRBC25, lines 4660–4670]. Thus $A_x^i = \delta_{i, 3x}$. Let $T^{i,j}$ be the bond-two CZX tensor in (160) of Ref. [FRBC25, lines 4505–4547], retaining both output Z gates and the unnormalized Hadamard weights. Contracting its input with block x gives

$$B_x^i = \sum_{j=0}^3 T^{i,j} A_x^j = \delta_{i, 3(1-x)} C_x, \quad (2)$$

$$C_0 = T^{3,0} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad C_1 = T^{0,3} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \quad (3)$$

Denote the source's printed left action tensor by p_x and its printed right action tensor by q_x . In the normalized convention $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, their coordinates are

$$p_0 = (0, -i), \quad p_1 = (1, 0), \quad (4)$$

$$q_0 = \frac{1}{2} \begin{pmatrix} -i \\ i \end{pmatrix}, \quad q_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (5)$$

These are the formulas printed in Ref. [FRBC25, lines 4690–4694]. The source does not explicitly specify the normalization of $|\pm\rangle$ there. The convention in (5) is consistent with the black factor in the tensor decomposition and the later defect evaluations [FRBC25, lines 4589–4601 and 4800–4859]. The left and right action symbols are ordinary row and column contractions; there is no extra matrix between them in the interior diagram.¹

Direct multiplication gives, for each x ,

$$C_x^2 = C_x, \quad p_x q_x = \frac{1}{2}, \quad (6)$$

$$p_x C_x q_x = 1, \quad C_x q_x p_x C_x = C_x. \quad (7)$$

For a nonempty word supported on the sole nonzero letter of B_x , its matrix product is C_x ; for any other nonempty word it is zero. Consequently (7) proves the interior identity at every positive length. At length zero, (6) gives 1/2 instead of 1. Changing the common normalization of the two printed kets cannot remove the conflict: replacing q_x by tq_x gives $p_x C_x(tq_x) = t$ and $p_x(tq_x) = t/2$. These two quantities cannot both equal one.

3 The valid exterior identity and a local replacement

For a word $\mathbf{a}$, let $B_x^{\mathbf{a}}$ and $A_{1-x}^{\mathbf{a}}$ denote the ordered products of the corresponding letters. The exterior action identity (59) in Ref. [FRBC25, lines 1788–1838] has two buffers of length $m \geq 1$ after the stipulated blocking and a middle word of length $n + 1 \geq 1$. The printed pair satisfies precisely that identity:

$$B_x^{\mathbf{a}} q_x A_{1-x}^{\mathbf{c}} p_x B_x^{\mathbf{b}} = B_x^{\mathbf{acb}}, \quad |\mathbf{a}| = |\mathbf{b}| \geq 1, \quad |\mathbf{c}| \geq 1. \quad (8)$$

If any letter is unsupported, both sides vanish. Otherwise the left side is $C_x q_x p_x C_x = C_x$, which is also the right side.

A local replacement of the left boundary gives an all-word reduction. Set

$$V_x = p_x C_x, \quad W_x = q_x, \quad (9)$$

$$V_0 = (i, -i), \quad V_1 = (1, 1). \quad (10)$$

¹The archived drawing definitions are in `Papers/2502.20257/Commands.tex`, lines 372–399.

Then $V_x W_x = p_x C_x q_x = 1$. Idempotency of C_x gives, for every word,

$$V_x B_x^{\mathbf{a}} W_x = A_{1-x}^{\mathbf{a}} \quad (|\mathbf{a}| \geq 0). \quad (11)$$

For the empty word this is $V_x W_x = 1$; for nonempty words it follows from the same supported-letter calculation. Moreover,

$$V_x B_x^{\mathbf{b}} = p_x B_x^{\mathbf{b}} \quad (|\mathbf{b}| \geq 1). \quad (12)$$

Thus replacing p_x by V_x leaves (8) unchanged. This replacement is not a scalar gauge transformation: the rows in (10) are not scalar multiples of the corresponding rows in (4). In particular, (12) does not permit substitution in an unbuffered defect formula.

4 Verdict and formal boundary

For the source's actual nonzero injective GHZ sectors, the printed CZX pair fails the interior identity at length zero, which the source explicitly includes by $n \geq 0$ [FRBC25, line 1870]. This nondegenerate counterexample warrants the false-source classification for the specific assertion that the printed pair satisfies the all-length interior identity, not for general existence of action tensors or later defect claims. Its nonempty interior identities and buffered exterior identities remain valid. A different, dressed pair satisfies the genuine all-word reduction relation without an extra hypothesis. The general action-tensor audit retains that all-word relation; this calculation does not weaken it.²

The later defect evaluations are not invalidated by this empty-word calculation; this is not a certificate of their global compatibility. The source gives $g[0] = |00\rangle$ and $g[1] = -i|11\rangle$ [FRBC25, lines 4800–4859]. These evaluations use the literal printed action tensors, not the replacement (9). Their existing derivation remains separate, as does the normalization issue for the different 2017 printed CZX tensor.³ No correction of fusion tensors, defect vectors, or the global symmetry is asserted here.

The finite identities, the nonempty printed interior and exterior identities, the failure of the printed empty contraction, and the all-word dressed reduction with exterior buffer length one are formally verified. The discrepancy and the separate all-word replacement are resolved; the printed pair itself still fails at length zero.⁴

²See `docs/paper-gaps/fbc25_mpu_action_stabilizer_source_audit.tex`, particularly its discussion of the interior identity.

³The respective notes are `docs/paper-gaps/fbc25_czx_defect_domains.tex` and `docs/paper-gaps/mpu_czx_tensor_normalization.tex`.

⁴In `TNLean/MPS/MPDO/CZXActionTensors`, `MPOTensor.CZX.printedAction_finite_identities`, `MPOTensor.CZX.printedAction_interior`, `MPOTensor.CZX.printedAction_exterior`, `MPOTensor.CZX.printedAction_empty_ne`, and `MPOTensor.CZX.printedAction_not_isReduction` certify the precise printed boundary. The separate correction is certified by `MPOTensor.CZX.dressedActionBra_absorption`, `MPOTensor.CZX.dressedAction_isReduction`, and `MPOTensor.CZX.dressedAction_isReductionExteriorBufferLength`.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.