

Circle and Nonzero-Complex Coefficients in Degree-Two Cohomology

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Verdict: clarification (resolved).

1 Source context

For a finite on-site symmetry group G , Ref. [FRBC25, lines 1927–1931] labels the fully unbroken phases by $H^2(G, \mathbb{C}^\times)$, or equivalently by $H^2(G, U(1))$. Its footnote regards the two coefficient groups as topological abelian groups and invokes an induced comparison in sheaf cohomology over a CW complex.

The formal development instead uses algebraic group cohomology. It regards G as finite and discrete and computes cohomology from inhomogeneous cochains with trivial coefficient action. The sheaf-cohomology sentence does not directly identify these algebraic cochain complexes.

2 Finite-discrete comparison

Let $i : U(1) \longrightarrow \mathbb{C}^\times$ be inclusion and let $q : \mathbb{C}^\times \longrightarrow U(1)$ be the phase homomorphism $q(z) = z/|z|$. Then

$$q \circ i = \text{id}_{U(1)}, \quad \frac{z}{i(q(z))} = |z| \in \mathbb{R}_{>0}. \quad (1)$$

Consequently, if ω is a $\mathbb{C}^\times$ -valued 2-cocycle, then $q \circ \omega$ is a $U(1)$ -valued 2-cocycle and its radial residual is

$$\frac{\omega(g, h)}{i(q(\omega(g, h)))} = |\omega(g, h)|. \quad (2)$$

The right-hand side of (2) is a positive generalized cocycle on the one-point block space. The finite-group positive-cocycle theorem gives a positive 1-cochain χ such that

$$|\omega(g, h)| = \frac{\chi(g)\chi(h)}{\chi(gh)}. \quad (3)$$

Thus ω and $i \circ q \circ \omega$ are cohomologous. Every $\mathbb{C}^\times$ -valued class therefore has a unit-modulus representative.

Equation (1) also shows, without assuming that G is finite, that the map induced by i is injective: the map induced by q is its left inverse. Combining this injectivity with the finite-group surjectivity above gives

$$H^2(G, U(1)) \cong H^2(G, \mathbb{C}^\times) \quad (G \text{ finite}), \quad (4)$$

for algebraic group cohomology with trivial coefficient actions.

3 Formal treatment

The formal proof implements the phase map, the polar identity, the positive one-block trivialization, and the induced maps on the canonical degree-two group cohomology object.¹

Verdict: resolved clarification. The source’s finite-group equivalence is available for the algebraic cohomology used in the classification, but it is established here by a direct polar-decomposition argument rather than by identifying it with the sheaf cohomology comparison in the footnote.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.

¹Formal declarations: `TNLean.Algebra.ScalarCocycle.cohomologousTo_circlePhaseInclusion` and `TNLean.Algebra.circleH2EquivComplexUnitsH2` in `TNLean/Algebra/CircleCohomology`.