

Reciprocal Factors in the Block-Independence Argument

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Verdict: local-correction (resolved).

1 The gauge multiplier

Let G act on a set X , and let $L: X \times G^2 \rightarrow \mathbb{C}^\times$ be an L -symbol. A fusion scalar $\beta_{g,h}$ and an action scalar γ_g^x act by multiplication with

$$M_{g,h}^x(\beta, \gamma) = \frac{\gamma_{gh}^x \beta_{g,h}}{\gamma_g^{hx} \gamma_h^x}. \quad (1)$$

At line 6990, Ref. [FRBC25] instead repeats γ_g in the second denominator factor. The second factor must be γ_h^x , as in (1); its group index is the second argument of $L_{g,h}^x$.

2 Factorization and the relative character

The source defines

$$\ell_{a,b;r}^x = \frac{L_{ar,r^{-1}b}^x}{L_{a,b}^x}. \quad (2)$$

Taking $a = g$, $b = h$, and $r = h$ in (2) gives

$$\ell_{g,h;h}^x = \frac{L_{gh,e}^x}{L_{g,h}^x}, \quad (3)$$

$$L_{g,h}^x = \frac{L_{gh,e}^x}{\ell_{g,h;h}^x}. \quad (4)$$

Thus line 6996 of Ref. [FRBC25] must divide by $\ell_{g,h;h}^x$ rather than multiply by it. Compatibility at (gh, e, e) gives

$$L_{gh,e}^x = \frac{L_{e,e}^x}{\omega(gh, e, e)}, \quad (5)$$

$$L_{g,h}^x = \frac{L_{e,e}^x}{\omega(gh, e, e) \ell_{g,h;h}^x}. \quad (6)$$

Under block independence, $\ell_{g,h;h}^x$ may be evaluated at any fixed $x_0 \in \mathbf{X}$. Hence the corrected block factor is

$$A(g, h) = (\omega(gh, e, e) \ell_{g,h;h}^{x_0})^{-1}, \quad (7)$$

$$L_{g,h}^x = A(g, h) L_{e,e}^x. \quad (8)$$

These identities are formalized directly from the definition of ℓ and compatibility.¹

Compatibility at (e, e, g) also gives

$$L_{e,g}^x = \omega(e, e, g) L_{e,e}^{gx}. \quad (9)$$

Combining (6) at (e, g) with (9) yields

$$L_{e,e}^{gx} = \rho(g) L_{e,e}^x, \quad (10)$$

$$\rho(g) = (\ell_{e,g;g}^{x_0} \omega(g, e, e) \omega(e, e, g))^{-1}. \quad (11)$$

The factor $\ell_{e,g;g}^{x_0}$ at line 7016 of Ref. [FRBC25] must therefore occur reciprocally. Applying (10) successively to h and g shows that

$$\rho(gh) = \rho(g) \rho(h). \quad (12)$$

No transitivity or finiteness assumption on the action is needed.²

3 Forward and inverse gauges

Define

$$\beta_{g,h}^+ = \frac{A(g, h)}{\rho(h)}, \quad (13)$$

$$\gamma_g^{+,x} = (L_{e,e}^{gx})^{-1}. \quad (14)$$

Using (10), their multiplier is

$$M_{g,h}^x(\beta^+, \gamma^+) = \frac{A(g, h)}{\rho(h)} \frac{(L_{e,e}^{ghx})^{-1}}{(L_{e,e}^{ghx})^{-1} (L_{e,e}^{hx})^{-1}} \quad (15)$$

$$= A(g, h) L_{e,e}^x \quad (16)$$

$$= L_{g,h}^x. \quad (17)$$

Consequently, (β^+, γ^+) sends the trivial L -symbol to L :

$$(\beta^+, \gamma^+) \triangleright 1 = L. \quad (18)$$

¹Formal declarations: `TNLean.Algebra.LSymbol.e11`, `TNLean.Algebra.LSymbol.blockFactor`, and `TNLean.Algebra.LSymbol.factorization_of_isBlockIndependentE11` in `TNLean/Algebra/LSymbolBlockIndependence`.

²Formal declarations: `TNLean.Algebra.LSymbol.relativeCharacter`, `TNLean.Algebra.LSymbol.relativeCharacter_mul_base`, and `TNLean.Algebra.LSymbol.relativeCharacter_mul` in `TNLean/Algebra/LSymbolBlockIndependence`.

The displayed factor in the source is therefore a forward gauge multiplier, not a multiplier that trivializes L . The pointwise inverse gauges give the required conclusion:

$$((\beta^+)^{-1}, (\gamma^+)^{-1}) \triangleright L = 1. \quad (19)$$

For a supplied compatible scalar L -symbol, this proves the implication from block independence to gauge triviality and, together with the immediate converse implications, the four-way scalar equivalence used in the proof of Proposition `prop:technical01` of Ref. [FRBC25].³

4 Scope of the formal result

The formal declarations start with a supplied compatible scalar L -symbol and prove the algebraic equivalence among its four scalar gauge properties. The fourth condition of Proposition `prop:technical01` in Ref. [FRBC25] is instead stated for an MPS–MPU pair. A tensor-facing result must still construct the associated L -symbol and prove that the MPS and MPU tensor gauges induce the scalar multiplier (1). The scalar calculation does not supply that bridge and therefore does not by itself establish the full source proposition.

5 Conclusion

The four local corrections form one reciprocal chain. The denominator of the joint gauge multiplier uses γ_h^x ; the definition of ℓ forces division by $\ell_{g,h;h}^x$ in the factorization; the relative character consequently contains the reciprocal of $\ell_{e,g;g}^{x_0}$; and the resulting factor is a forward gauge multiplier whose inverse trivializes the supplied L -symbol. These reciprocal corrections are resolved. The tensor-facing bridge required for the full statement of Proposition `prop:technical01` remains outside the scalar result.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.

³Formal declarations: `TNLean.Algebra.LSymbol.factorFusionGauge`, `TNLean.Algebra.LSymbol.factorActionGauge`, `TNLean.Algebra.LSymbol.gauge_one_eq_of_isBlockIndependentEll`, `TNLean.Algebra.LSymbol.gauge_inv_eq_one_of_isBlockIndependentEll`, and `TNLean.Algebra.LSymbol.hasTrivialGauge_of_isBlockIndependentEll` in `TNLean/Algebra/LSymbolBlockIndependence`. The capstone equivalences are `TNLean.Algebra.LSymbol.hasTrivialGauge_iff_hasBlockConstantGauge`, `TNLean.Algebra.LSymbol.hasTrivialGauge_iff_hasTrivialEllGauge`, and `TNLean.Algebra.LSymbol.hasTrivialGauge_iff_isBlockIndependent`.