

Nondegeneracy in the Alternating-Network Proportionality Lemma

2026-08-30

Verdict: local-correction (resolved).

1 Local nondegeneracy convention

Lemma lemma:1 of Ref. [FRBC25, lines 2296–2395] considers a connected open chain

$$A_1 B_1 A_2 B_2 \cdots A_n B_n = A_1 B'_1 A_2 B'_2 \cdots A_n B'_n, \quad (1)$$

where every A_i is injective. It concludes that there are scalars β_i such that

$$\begin{aligned} B_i &= \beta_i B'_i, \\ \prod_{i=1}^n \beta_i &= 1. \end{aligned} \quad (2) \quad (3)$$

The formal treatment adopts the intended nondegenerate convention that every intervening tensor B_i is nonzero. This convention is present at both applications of the lemma in the article.

2 Nondegenerate correction

The corrected statement assumes

$$B_i \neq 0 \quad (1 \leq i \leq n). \quad (4)$$

Contracting the physical leg of each A_i against an inverse tensor, exactly as in the source proof, separates the intervening tensors and gives

$$\bigotimes_{i=1}^n B_i = \bigotimes_{i=1}^n B'_i. \quad (5)$$

Condition (4) makes the left-hand side of (5) nonzero. Proportionality of the factors of two equal nonzero pure tensors then gives (2)–(3). A one-sided non-vanishing assumption is sufficient: the pure-tensor equality forces every B'_i to be nonzero as well.

The two uses in Ref. [FRBC25] already lie in this nondegenerate regime. The truncated symmetry U_g^L is unitary by the statement at line 2099. For the application at line 2567, lines 2444–2466 identify one side of the top boundary network with $(U_{g^{-1}}^L)^\dagger$. This network is nonzero; hence none of its constituent intervening factors can vanish. One may take these factors as the single-sided family B_i in (4). For the application at line 3157, lines 3033–3053 identify the corresponding network with $\sigma_g(U_{g^{-1}}^L)^\dagger$, where $\sigma_g \in \{1, -1\}$. It is again nonzero for the same reason. Neither use requires a stronger invertibility assumption on every B_i .

3 Formal treatment

The formal theorem models the connected open chain with its horizontal ends uncontracted. Matrix units supplied to the one-sided inverses of the injective A_i retain both virtual indices, so their contraction proves (5) rather than merely an equality of matrix products. The final factorwise conclusion uses the existing pure-tensor proportionality theorem.¹

Verdict: resolved local fix. The intended nondegeneracy convention is present at both applications in the article. The unrestricted printed wording is not used as a formal theorem.

References

- [FRBC25] Adrián Franco Rubio, Arkadiusz Bochniak, and J. Ignacio Cirac. Symmetry defects and gauging for quantum states with matrix product unitary symmetries, 2025.

¹Formal declaration: `MPSTensor.exists_scalars_of_openAlternatingNetwork_eq` in `TNLean/MPS/Chain/AlternatingNetwork`.