

Trace Preservation in the Forward Direction of Theorem 4.1 (de las Cuevas–Cirac–Schuch–Pérez-García)

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Verdict: false-source (wip).

1 The assertion

This note concerns the forward implication in Theorem 4.1 of de las Cuevas, Cirac, Schuch, and Pérez-García [DICCSPG17, Theorem 4.1]. The relevant source passages are the definition of the irreducible forms at `Papers/1708.00029/main.tex:245--266` and `Papers/1708.00029/main.tex:313--331`, the definitions of divisibility and refinement at `Papers/1708.00029/main.tex:717--724`, the theorem at `Papers/1708.00029/main.tex:728--731`, and its forward proof at `Papers/1708.00029/main.tex:735--810`. The companion note `dccsp17_root_kraus_rank_thm41.tex` records the separate gap in the converse implication.

A tensor $B = (B^1, \dots, B^d)$ of bond dimension D is in *irreducible form* when

$$B^i = \bigoplus_{j \in J} \mu_j B_j^i, \quad \mu_j > 0, \quad (1)$$

with every block map $\mathcal{E}_{B_j}(X) = \sum_i B_j^i X (B_j^i)^\dagger$ irreducible of spectral radius one. It is in *irreducible form II* when moreover every $\mathcal{E}_{B_j}$ is trace preserving with a diagonal positive fixed point. The multiplicities μ_j are arbitrary positive scalars in both forms; the source normalizes the blocks, not the weights.

A trace-preserving completely positive map $\mathcal{E}$ is called *p-divisible* when there is a trace-preserving completely positive map $\mathcal{E}'$ with $\mathcal{E} = (\mathcal{E}')^p$. The family $\mathcal{V}(B)$ is *p-refinable* when there are a tensor $A = (A^1, \dots, A^d)$ and an isometry $W : \mathbb{C}^d \rightarrow (\mathbb{C}^d)^{\otimes p}$ with

$$|V_{pN}(A)\rangle = W^{\otimes N} |V_N(B)\rangle \quad \text{for every } N. \quad (2)$$

Theorem 4.1 asserts, for B in irreducible form II, that $\mathcal{V}(B)$ is *p-refinable* if and only if the transfer map $\mathcal{E}_B(X) = \sum_i B^i X (B^i)^\dagger$ is *p-divisible*.

2 The counterexample

Let B_0 be any tensor in irreducible form II whose multiplicities all equal one and whose family is p -refinable with witness (A_0, W) . The simplest instance has $d = D = 1$, $B_0 = (1)$, $A_0 = (1)$, and $W = 1$, for which $|V_{pN}(A_0)\rangle = 1 = |V_N(B_0)\rangle$ at every length. Fix $p \geq 1$ and $\lambda > 1$ and set

$$B := \lambda B_0, \quad A := \lambda^{1/p} A_0. \quad (3)$$

The tensor B is in irreducible form II: it has the blocks of B_0 and the multiplicities $\lambda \mu_j$, which are positive. The family $\mathcal{V}(B)$ is p -refinable with witness (A, W) , because

$$|V_{pN}(A)\rangle = \lambda^N |V_{pN}(A_0)\rangle = \lambda^N W^{\otimes N} |V_N(B_0)\rangle = W^{\otimes N} |V_N(B)\rangle. \quad (4)$$

Its transfer map is $\mathcal{E}_B = \lambda^2 \mathcal{E}_{B_0}$, so $\text{tr } \mathcal{E}_B(X) = \lambda^2 \text{tr } X$. A p -th power of a trace-preserving map is trace preserving, so $\mathcal{E}_B$ is not p -divisible. The forward implication of Theorem 4.1 therefore fails on nondegenerate data satisfying its printed hypothesis.

The concrete choice $p = 2$, $B_0 = A_0 = (1)$, and $\lambda = 4$ is formalized: $B = (4)$ is literally the positive-weight block $4(1)$, has the two-site refinement $A = (2)$, and has transfer map $x \mapsto 16x$. The formal theorem proves that B is in irreducible form, is 2-refinable, and is not 2-divisible as a channel.¹

The printed argument breaks at its last line, `Papers/1708.00029/main.tex:806--810`. It constructs $\tilde{A}$ with $\tilde{A}^{(p)} = C$ and concludes $\mathcal{E}_{\tilde{A}}^p = \mathcal{E}_C = \mathcal{E}_B$, which is correct, but $\tilde{A}$ is a unitary conjugate of $A' = \bigoplus_j \mu_j A'_j$, whose transfer map is trace preserving only when every multiplicity of A equals one. The argument never constrains the multiplicities of B , so it cannot deliver the trace preservation that the conclusion requires.

3 The corrected statement

Add to the hypothesis of the forward implication that $\mathcal{E}_B$ is trace preserving. Since

$$\sum_i (B^i)^\dagger B^i = \bigoplus_{j \in J} \mu_j^2 \mathbf{1}_{D_j} \quad (5)$$

for a tensor in irreducible form II, this is the same as requiring every multiplicity of B to equal one. The added hypothesis is presupposed by the source: its definition of p -divisibility applies only to trace-preserving maps, so the conclusion “ $\mathcal{E}_B$ is p -divisible” already asserts trace preservation of $\mathcal{E}_B$.

¹See `MPSTensor.refinement_normalization_counterexample` and `MPSTensor.refinement_ScalarCounterexample` in `TNLean/MPS/Periodic/Symmetry/RefinementNormalizationCounterexample.lean`. At $p = 1$ every tensor is 1-refinable, while 1-divisibility is exactly the channel condition; see `MPSTensor.isPRefinable_one` and `MPSTensor.isPDivisibleChannel_one_iff` in `TNLean/MPS/Periodic/Symmetry/Theorem41OneSite.lean`.

With this hypothesis the printed argument closes. The tensor $C^{i_1 \dots i_p} = \sum_i W^{(i_1 \dots i_p), i} B^i$ has the blocks and multiplicities of B , all equal to one. The equal-case theorem [DICCSPG17, Theorem 3.8] matches the blocks of $A^{(p)}$ with those of C and relates their multiplicities by roots of unity after reordering. The blocks of $A^{(p)}$ arising from a block A_j of A carry the multiplicity μ_j^p , so μ_j^p is a root of unity, and $\mu_j > 0$ forces $\mu_j = 1$. Hence A' and $\tilde{A} = U^\dagger A' U$ are left canonical, $\mathcal{E}_{\tilde{A}}$ is trace preserving, and $\mathcal{E}_B = \mathcal{E}_{\tilde{A}}^p$ exhibits p -divisibility. The converse implication is unaffected, since it assumes p -divisibility and therefore trace preservation of $\mathcal{E}_B$ from the start.

4 The formal statement

The formal irreducible form is a statement about generated families rather than about the matrices of the tensor: a tensor is in irreducible form when it generates, at every length including zero, the same matrix product vectors as a scalar-weighted direct sum of periodic left-canonical blocks with positive weights.² It does not fix the transfer map of the tensor. Two consequences follow for the forward canonicalization hypothesis, which asks, for every tensor B in this formal irreducible form whose family is p -refinable, for a left-canonical tensor $\tilde{A}$ with $\mathcal{E}_B = \mathcal{E}_{\tilde{A}^{[p]}}$.³

First, the scaling counterexample refutes the hypothesis at bond dimension one for every positive physical dimension and every $p \geq 1$: the tensor λB_0 generates the same family as the weighted block sum with weight λ , so it is in the formal irreducible form, and the conclusion would identify the map $x \mapsto \lambda^2 x$ with a p -th power of the identity.

Second, even after adding trace preservation of $\mathcal{E}_B$, the formal irreducible form does not supply the block structure that the printed argument uses. For a left-canonical normal tensor B_0 and an invertible Y , the tensor $Y B_0 Y^{-1}$ generates the same family as B_0 and is in the formal irreducible form, while $\mathcal{E}_{Y B_0 Y^{-1}}$ is trace preserving exactly when $Y^\dagger Y$ is a fixed point of $\mathcal{E}_{B_0}^*$, that is, when Y is a scalar multiple of a unitary. Recovering a unitary similarity to a block form from equality of families and trace preservation is a statement about arbitrary tensors that the cited argument does not contain; the source instead takes B literally in the form (1).

The conditional formal theorems remain correct, but their hypothesis cannot be discharged. The elimination plan is to restate the forward direction for a tensor given as a multiplicity-bearing block form with every multiplicity equal to one, the structure already used by the formal equal-case theorem,⁴ and then to follow the printed argument. The single-block ingredients now have formal coun-

²The definition is `MPSTensor.IsIrreducibleForm` in `TNLean/MPS/Periodic/Defs.lean`.

³The hypothesis is `MPSTensor.PRefinementCanonicalization` and the conditional forward theorem is `MPSTensor.thm_4_1_p_refinement_forward`, both in `TNLean/MPS/Periodic/Symmetry/Theorem41Forward.lean`.

⁴The equal-case theorem is `MPSTensor.fundamentalTheorem_periodic_equalCase_irreducibleForm` in `TNLean/MPS/Periodic/EqualCaseGlobal.lean`.

terparts: `MPSTensor.IsPeriodic.exists_stepOrbit_blockDecomposition` constructs the prescribed periodic blocks and their cyclic projections; `MPSTensor.exists_isLeftCanonical_blockTensor_eq_of_orbit_action` redistributes the orbit phases; and `MPSTensor.exists_isLeftCanonical_blockTensor_eq_of_unitary_conj` performs the final unitary conjugation. Moreover, `MPSTensor.IsPeriodic.exists_unitary_stepOrbit_decomposition` assembles the orbit isometries into one unitary retaining their original inclusions, and transports every admissible orbit phase through those same coordinates to a left-canonical blocked root. These correspond to `Papers/1708.00029/main.tex:432--456` and `Papers/1708.00029/main.tex:765--810`. The arbitrary-input normalization is now supplied by `MPSTensor.exists_unitWeight_periodic_presentation_of_blocked_sameMPVPos`: when a positive blocking matches a periodic decomposition with unit-modulus multiplicities, the original tensor admits a unit-weight periodic presentation. The exact target bond dimension follows from the equal-case theorem, without requiring equality at length zero or assuming an initial canonical form. These results are assembled over the full multiplicity-bearing decomposition in `TNLean/MPS/Periodic/Symmetry/Theorem41ForwardBlockForm.lean`: the equal-case gauge is identified on the original orbit supports through `MPSTensor.exists_unitary_matching_of_periodic_block_families`, and `MPSTensor.isPDivisibleChannel_of_isPRefinable_toTensorFromBlocks` proves the corrected forward implication for a direct sum of left-canonical periodic blocks with unit-modulus multiplicities, with no further hypothesis. What remains, tracked in issue #7762, a child of issue #622, is to retire the conditional formal theorems below in favour of this statement. Until then, every formal theorem whose proof consumes a refuted hypothesis carries an unfaithfulness marker naming this note: the conditional forward theorem, its blocked-stage variant, the bundled biconditional, and the three reduction theorems between the refuted hypotheses, namely `MPSTensor.peripheralEqualCase_periodicFT_of_sameMPV`, `MPSTensor.pRefinementCanonicalization_of_peripheralEqualCase_periodicFT_of_sameMPV`, and `MPSTensor.peripheralEqualCaseRootFromZGauge_of_rootChannel`. The hypotheses themselves are `MPSTensor.PeripheralEqualCaseRootChannelOfZGauge`, `MPSTensor.PeripheralEqualCaseRootFromZGauge`, `MPSTensor.PeripheralEqualCasePeriodicFTOfSameMPV`, and `MPSTensor.PRefinementCanonicalization`. The first implies the second, the second together with the blocked Z -gauge extraction hypothesis `MPSTensor.PeripheralEqualCaseZGaugeOfSameMPV` implies the third, and the third implies the fourth. For $p \geq 1$ each of the four is refuted by the one-dimensional data $C = A^{(p)} = (\lambda)$ with the trivial Z -gauge, since no channel has p -th power $x \mapsto \lambda^2 x$.

5 Conclusion

The forward implication of Theorem 4.1 is false as printed: a positive rescaling of any p -refinable tensor in irreducible form II is again in irreducible form II

and p -refinable, while its transfer map is not trace preserving and hence not p -divisible. The corrected statement adds trace preservation of $\mathcal{E}_B$, equivalently unit multiplicities, which the source's definition of p -divisibility presupposes; the printed argument then proves it. The formal forward canonicalization hypothesis inherits the counterexample and is refuted independently by non-unitary similarities, so it must be replaced by a restatement over block-form tensors with trace-preserving transfer map rather than discharged.

References

- [DICCSPG17] Gemma De las Cuevas, J. Ignacio Cirac, Norbert Schuch, and David Pérez-García. Irreducible forms of matrix product states: Theory and applications. *Journal of Mathematical Physics*, 58(12):121901, 2017.