The Root-Channel Kraus-Rank Step in Theorem 4.1

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Verdict: unfaithful (open).

1 The assertion

The reverse implication in Theorem 4.1 of de las Cuevas, Cirac, Schuch, and Pérez-García is the point at issue [DlCCSPG17, Theorem 4.1].[1] The source passages are `Papers/1708.00029/main.tex:717--731` and `Papers/1708.00029/main.tex:812--818`.

Let $B = (B^1, \dots, B^d)$ be an MPS tensor, let $p \geq 1$, and let $\mathcal{E}_B$ be the transfer channel

$$\mathcal{E}_B(X) = \sum_{i=1}^{d} B^i X (B^i)^\dagger.$$

The paper says that a trace-preserving completely positive map $\mathcal{E}$ is p-divisible if there is another trace-preserving completely positive map $\mathcal{E}'$ with

$$\mathcal{E} = (\mathcal{E}')^p.$$

This is a channel-level condition. It places no restriction on the number of Kraus operators needed to represent the root channel $\mathcal{E}'$.

In the same passage, B is called p-refinable if there are a tensor $A = (A^1, \dots, A^d)$ with the same physical alphabet and an isometry

$$W : \mathbb{C}^d \longrightarrow (\mathbb{C}^d)^{\otimes p}$$

such that

$$|V_{pN}(A)\rangle = W^{\otimes N} |V_N(B)\rangle \qquad \text{for every } N.$$

Thus the tensor root in a refinement is indexed by the original d physical letters. The theorem states the unconditional equivalence

$$B \text{ is } p\text{-refinable} \quad \Longleftrightarrow \quad \mathcal{E}_B \text{ is } p\text{-divisible}$$

for B in irreducible form II.

[1] For traceability, this note corresponds to issue #1046. Related formal work is recorded in issue #670, issue #821, issue #622, and issue #664.

2 The point at issue

The reverse proof begins with p-divisibility of $\mathcal{E}_B$. This gives a trace-preserving completely positive map $\mathcal{E}'$ such that $\mathcal{E}_B = (\mathcal{E}')^p$. The proof then writes that there exists a tensor A such that

$$\mathcal{E}_B = \mathcal{E}_A^p,$$

and proceeds to compare two Kraus representations of the same channel by the usual isometric freedom of Kraus representations.

The displayed passage is the missing step. To realize the channel root $\mathcal{E}'$ as $\mathcal{E}_A$ for a tensor $A = (A^1, \ldots, A^d)$, the root channel must admit a Kraus representation with at most d Kraus operators. Equivalently, its minimal Kraus rank, or Choi rank, must be at most d. If the root channel has minimal Kraus rank $r > d$, then it can be represented by an r-letter tensor but not by a d-letter tensor. The resulting isometry would land in $(\mathbb{C}^r)^{\otimes p}$ rather than in the space $(\mathbb{C}^d)^{\otimes p}$ required by the definition of p-refinement.

This is a mathematical point, not a matter of notation. A completely positive map can require more than a prescribed number of Kraus operators. The channel-level assertion $\mathcal{E}_B = (\mathcal{E}')^p$ does not by itself rule out the possibility that every Kraus representation of $\mathcal{E}'$ has more than d terms. Therefore one needs either an argument that the special hypotheses of Theorem 4.1 force a d-term Kraus representation for some root channel, or a different proof of the reverse implication.

3 The formal statement

The formal definition of p-divisibility follows the channel-level definition from the paper: it asserts the existence of a trace-preserving completely positive root channel and does not include any bound on the root channel's Kraus rank. The conditional reverse theorem therefore separates the missing step from the rest of the proof. [2]

The extra formal hypothesis is an inverse canonicalization statement. In ordinary mathematical terms, it says that whenever B is in irreducible form II and $\mathcal{E}_B$ is p-divisible, there is a d-letter tensor A such that

$$\mathcal{E}_B = \mathcal{E}_{A^{[p]}}.$$

If such a tensor-level root is available, the rest of the argument follows the source proof: the tensor $A^{[p]}$ obtained by blocking and the original tensor B give two finite Kraus representations of the same completely positive map, and isometric

[2] The definition is `MPSTensor.IsPDivisibleChannel` in `TNLean/MPS/Periodic/Symmetry/Theorem41Defs.lean:24--32`. The conditional reverse theorem is `MPSTensor.thm_4_1_p_refinement_reverse` in `TNLean/MPS/Periodic/Symmetry/Theorem41Reverse.lean:120--142`.

freedom of Kraus representations gives the isometry $W : \mathbb{C}^d \to (\mathbb{C}^d)^{\otimes p}$ needed for refinement. [3]

The formal theorem also isolates a sufficient channel-theoretic hypothesis for the missing step: if every trace-preserving completely positive root $\mathcal{E}'$ with $\mathcal{E}_B = (\mathcal{E}')^p$ has Kraus rank at most d, then one can choose a d-term Kraus representation by padding with zero Kraus operators and obtain the required tensor A. This is a genuine mathematical hypothesis about the root channel, not a change of notation. The minimal-Kraus fact that the least number of Kraus operators equals the Choi rank is the right background theorem, but it does not itself prove the required inequality $\mathrm{rank}_{\mathrm{Kraus}}(\mathcal{E}') \leq d$ for a root of $\mathcal{E}_B$. [4]

4 Conclusion

The formal theorem adds a hypothesis relative to the source theorem. The source statement claims that channel-level p-divisibility of $\mathcal{E}_B$ is already enough to construct a p-refinement of B, but the reverse proof passes through a tensor root whose existence requires a Kraus-rank bound on the channel root. That bound is not part of the paper's definition of p-divisibility and is not proved in the cited converse paragraph.

The point left open is whether the channel root can always be chosen with Kraus rank at most d under the irreducible-form hypotheses of the theorem. The formal theorem therefore proves the reverse implication under an explicit inverse canonicalization hypothesis, or under the stronger root-Kraus-rank assumption which implies it. Until a paper-level argument proves such a bound, the reverse implication of Theorem 4.1 should be treated in the formal statement as conditional on that additional mathematical input.

References

[DlCCSPG17] Gemma De las Cuevas, J. Ignacio Cirac, Norbert Schuch, and David Pérez-García. Irreducible forms of matrix product states: Theory and applications. *Journal of Mathematical Physics*, 58(12):121901, 2017.

[3] The inverse canonicalization hypothesis is `MPSTensor.PRefinementInverseCanonicalization` in `TNLean/MPS/Periodic/Symmetry/Theorem41Reverse.lean:26--42`. The blueprint records the bundled conditional equivalence and the reverse conditional implication in `blueprint/src/chapter/ch22_periodic_ft.tex:2012--2042` and `blueprint/src/chapter/ch22_periodic_ft.tex:2096--2122`.

[4] The bounded-root formulation is `MPSTensor.PRefinementInverseRootKrausRankBound`; the implication from this bound to inverse canonicalization is `MPSTensor.pRefinementInverseCanonicalization_of_rootKrausRankBound`, recorded in `TNLean/MPS/Periodic/Symmetry/Theorem41Reverse.lean:98--118`.