Proof-Route Alignment for the Periodic Overlap Dichotomy (de las Cuevas–Cirac–Schuch–Pérez-García)

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Verdict: scope-restriction (open).

1 Scope

This note records, in mathematical terms, where the formal development of the periodic overlap dichotomy chooses a proof route that differs from the appendix argument of [1, Appendix A], and where its statements restrict the source.[1]

The source proposition is *equal-or-orthogonal-generalized*.[2] None of the deviations below is unfaithful in the smuggled-hypothesis sense: each formal statement is a faithful (or explicitly restricted) version of the source, and the route substitutions are mathematically equivalent.

2 Positive chain lengths

The matrix-product-vector equalities used by the periodic overlap argument are now stated for $N \geq 1$. In particular, `peripheralProportionalCase_periodicFT_of_sameMPVPos` assumes equality at every positive length, and `PeripheralProportionalCaseRootFromRescaling` concludes equality at every positive length after phase rescaling. This is sufficient because the overlap argument is asymptotic and uses only a tail of positive lengths. The empty word carries only a bond-dimension trace and plays no role in [1, theorem `thm:bd`, lines 613–623]. Likewise, propagation of nonzero cyclic-sector dimensions is derived directly from the corner-algebra equivalences and the cyclic projector shift, rather than from the empty-word trace identity.

[1] Formal files: `TNLean/MPS/Periodic/Overlap/`. Local source: `Papers/1708.00029/main.tex`. Paper-level umbrella: issue #619; proposition-level subtracking: issue #81; the Case-3 contraction and phase construction was tracked by issue #873 and is now formalized by `sectorTensor_proportional_of_blockedMatch`.

[2] Stated in [1], lines 589–609, and proved in Appendix A, lines 903–1118.

3 Different-period decay via the peripheral spectrum

Source.[3] For periods $m_a \neq m_b$, the paper blocks at $p = \mathrm{lcm}(m_a, m_b)$ and uses Lemma *bdcf*: the blocks $P_u A^{(p)}$ and $Q_v B^{(p)}$ are non-repeated normal tensors. Applying the one-site translation operator to $Q_v B^{(p)}$ versus $Q_{v+1} B^{(p)}$ yields a contradiction unless $\lim_N |\langle V_N(A)|V_N(B)\rangle| = 0$.

Lean route.[4] The same obstruction is realized spectrally. For a tensor A, write $\mathcal{E}_A$ and $\mathcal{E}_A^*$ for the transfer map and its adjoint, defined by

$$\mathcal{E}_A(Y) = \sum_i A^i Y A^{i\dagger}, \qquad \mathcal{E}_A^*(Y) = \sum_i A^{i\dagger} Y A^i. \tag{1}$$

If $D_a = D_b$ and $B^i = \zeta\, X A^i X^{-1}$, the comparison is

$$\mathcal{E}_B(Y) = \zeta\bar{\zeta}\, X \mathcal{E}_A(X^{-1} Y X^{-\dagger}) X^{\dagger}. \tag{Case 1}$$

Eigenvalue uniqueness for an irreducible completely positive map ([3], Theorem 6.3) forces $|\zeta| = 1$, so $\mathcal{E}_A$ and $\mathcal{E}_B$ have equal peripheral spectra. An m-periodic tensor has peripheral spectrum $\{e^{2\pi i r/m} : 0 \leq r < m\}$, so equality forces $m_a = m_b$. Thus different periods exclude gauge-phase equivalence, and *equal-or-orthogonal* (the irreducible trace-preserving overlap dichotomy) gives the decay. This substitutes the paper's lcm-blocking + translation argument; it is complete and proved.

4 Sector non-repetition via the blocked fixed-point structure

Source.[5] For a periodic block, $\mathcal{E}_A$ is irreducible with peripheral spectrum $\{\omega^r\}$, $\omega = e^{2\pi i/m}$. The blocked map $\mathcal{E}_A^m$ then has 1 as its only modulus-one eigenvalue (multiplicity m), with fixed-point set exactly $\{P_u \Lambda_A P_u\}_u$, while the adjoint $\mathcal{E}_A^{*m}$ has fixed points $\{P_u\}_u$. If, for $u \neq v$, a gauge-phase equivalence $C_u^{\mathbf{i}} = e^{i\xi} U C_v^{\mathbf{i}} U^{\dagger}$ held with $U = P_u U P_v$ ($UU^{\dagger} = P_u$, $U^{\dagger}U = P_v$), then

$$\mathcal{E}_A^m(U) = \sum_{\mathbf{i}} C_u^{\mathbf{i}} U C_v^{\mathbf{i}\dagger} \tag{2}$$

$$= e^{i\xi} U \sum_{\mathbf{i}} C_v^{\mathbf{i}} C_v^{\mathbf{i}\dagger} \tag{3}$$

$$= e^{i\xi} U. \tag{$\mathcal{E}_{C_v}(P_v) = P_v$}$$

so U is a modulus-one eigenvector of $\mathcal{E}_A^m$. As the only such eigenvalue is 1 with the diagonal fixed points $\{P_w \Lambda_A P_w\}$ and $U = P_u U P_v$ is off-diagonal for $u \neq v$, this is a contradiction.

[3] [1], Appendix A, lines 917–950.

[4] Formal declaration: `periodicOverlap_tendsto_zero_of_ne_period` in `DifferentPeriod.lean`.

[5] Lemma *bdcf* in [1], lines 404–423.

Lean status.[6] The lemma statement is the faithful non-repetition fact: distinct cyclic sectors (orthogonal projectors $P_u P_v = 0$) are not gauge-phase equivalent. The formal proof now follows the spectral route above. From a gauge-phase equivalence between two compressed sectors, it constructs a nonzero off-diagonal eigenvector of $\mathcal{E}_A^m$, using the sector corner-letter identity and the rectangular support isometries for the compressed corners. The existing off-diagonal vanishing lemma then gives the contradiction. The scalar normalization reuses the same Perron–Frobenius eigenvalue-uniqueness input as the different-period proof.[7]

Signature correction (periodicity is load-bearing). An adversarial proof-attempt found that the lemma was *underspecified*: with only the bare cyclic-projection data (orthogonal P_k summing to 1, the adjoint shift $\mathcal{E}_A^*(P_{k+1}) = P_k$, blocked-letter commutation, the trace formula, and orthogonality), the conclusion is *false* — a reducible $A = B \oplus B$ (for an irreducible period-m block B) admits the same cyclic-projection data yet has gauge-phase-equivalent distinct sectors. The spectral argument genuinely requires A to be a periodic (irreducible) block: that is what makes $\mathcal{E}_A$ irreducible with the stated peripheral spectrum and the sectors primitive. The hypothesis `hP:IsPeriodicmA` was therefore added to the lemma signature (it is supplied at the unique call site `sectorBlocks_not_gaugePhaseEquiv_of_ne`). With this hypothesis and the corner-isometry data, the spectral proof is now formalized.

5 Sector-match case: propagation and the phase construction

Source.[8] Given one matched pair, applying the translation operator T^l to the blocked equation *eq:Nm* and invoking Theorem 2.10 of [2] produces, at each offset, a phase and a unitary $U_{\tilde{v}+l} = P_{\tilde{u}+l} U_{\tilde{v}+l} Q_{\tilde{v}+l}$, so the offset $v - u = q$ is constant. Then each blocked product F_u becomes injective after N_0 repetitions, with a right inverse Ω_u; contracting the concatenated products with the Ω_u recovers per-site proportionality $A_u^i = \kappa_v\, e^{i\eta/m}\, B_v^i$. The phase bookkeeping is load-bearing: $\prod_v \kappa_v = 1$ (*eq:prodkappaprop*) and $|\kappa_v| = 1$ from $\| \sum_i A_u^{i\dagger} A_u^i \| = 1$, so $\kappa_v = e^{i\theta_v}$ with $\sum_v \theta_v = 0$; choosing ϕ_v with $\theta_v = \phi_v - \phi_{v+1}$ telescopes the per-sector phases into a single global phase $\xi = \eta/m$ and a single global unitary $U = \sum_u e^{i\phi_{u+q}} P_u U_{u+q} Q_{u+q}$, giving $A^i = e^{i\xi} U B^i U^\dagger$.

Lean status.[9] The statements track this two-stage structure: the T^l + Theorem 2.10 staircase, followed by the Ω_u contraction and the $\kappa/\theta/\phi$ phase

[6] Formal declaration: `not_gaugePhaseEquiv_of_orthogonal_cyclicSector_traces` in `SelfOverlap.lean`.

[7] Formal declarations: `exists_offDiag_eigenvector_of_gaugePhase_cast_left`, `offDiag_eigenvector_eq_zero_of_isPeriodic`, and `period_eq_of_gaugePhaseEquiv_of_isPeriodic`.

[8] [1], Appendix A, lines 961–1117; the initial blocked equation *eq:Nm* appears at lines 978–984.

[9] Formal declarations: `sectorMatch_propagation` in `SectorMatch/Propagation.lean`, and `sectorTensor_proportional_of_blockedMatch` in `SectorMatch/Contraction.lean`, and `periodicOverlap_gaugeEquiv_of_sector_match` in `SectorMatch/Consequences.lean`.

construction. The top-level theorem now invokes these component lemmas; the contraction leaf `sectorTensor_proportional_of_blockedMatch` and the resulting unitary global gauge are proved, and the repeated-blocks theorem follows. The docstrings have been realigned to cite the source equation labels and line ranges (replacing earlier published-version labels such as “A.8” and “A.14–A.18” that do not occur in the local source), and the $\kappa/\theta/\phi$ phase construction is now stated explicitly as load-bearing.

Formalized single-site inputs. The single-site off-diagonal decomposition is now formalized from the cyclic adjoint-shift relation.[10] It gives $P_{u+1}A^i = A^i P_u$ and $A^i = \sum_u P_{u+1}A^i P_u$ for a chosen cyclic sector decomposition. This is the inverse-indexed form of *eq:Aoffdiag*, lines 335–339; the later *eq:Auprop*, line 1014, defines the sector letters $A^i_u = P_u A^i P_{u+1}$.

The construction of the cyclic sectors also now records the letter-level corner identity and a rectangular support isometry for each corner.[11] If φ_u is the compression isomorphism from the compressed sector algebra onto $P_u M_D P_u$, then

$$\varphi_u(C_u^{\mathbf{i}}) = P_u(A^{(m)})^{\mathbf{i}} P_u.$$

This is the exposed formal version of the blocked corner-letter relation in Lemma *bdcf* and *Cu*. Together with the support isometry, it transports the compressed mixed-transfer eigenvector to the ambient corner and supplies the off-diagonal modulus-one eigenvector used in the spectral contradiction.

Let $O_{C,D}(N) := \langle V_N(C) \mid V_N(D)\rangle$. The one-site sector-overlap translation is also formalized.[12] It proves, for every $N > 0$,

$$O_{A_{u+1},B_{v+1}}(N) = O_{A_u,B_v}(N).$$

It is the overlap-sum form of the translation step in lines 985–1002, using the single-site relations inside the paired trace sum and the repository’s inverse cyclic indexing convention.

Formalized Case-3 contraction. After these single-site inputs, the contraction and gauge construction proceeds as follows:

- The m-factor cyclic Ω_u-contraction with the η bookkeeping: starting from the common right inverses for the injective tensors F_u, contract the cyclic product in lines 1041–1068 to obtain the uniform product-tensor identity *eq:resultprop*. The displayed inverse list in lines 1049–1052 has only $m-1$ factors, but the displayed concatenation, the $(mN_0 + 1)$ repetition count, and the definition of η_v require one inverse for each of the m inserted F-factors.

[10] The formal declarations are `offDiag_shift_of_adjoint_cyclic_shift`, `eq_sum_offDiag_of_adjoint_cyclic_shift`, `IsCyclicSectorDecomp.offDiag_shift`, and `IsCyclicSectorDecomp.eq_sum_offDiag`.

[11] Formal declarations: `exists_cyclic_sector_decomp_with_letter_and_isometry_after_blocking_of_isPeriodic` and its projected wrapper `exists_cyclic_sector_decomp_with_letter_after_blocking_of_isPeriodic`.

[12] Formal declaration: `sectorOverlap_succ_eq_of_cyclicSectorDecomp`.

- The final normalization and gauge construction: use the proved scalar extraction theorem to obtain κ_v with $\prod_v \kappa_v = 1$, prove $|\kappa_v| = 1$ from the left-canonical normalization, apply the finite-cycle coboundary lemma to write $\kappa_v = e^{i(\phi_v - \phi_{v+1})}$, and assemble the global unitary $U = \sum_u e^{i\phi_{u+q}} P_u U_{u+q} Q_{u+q}$.

These steps discharge the contraction obligation for the repeated-blocks theorem.[13] The tensor-product scalar extraction and product-one normalization have already been isolated in `PiTensorProductPhase.exists_kappa_of_piTensorProduct_eq_smul` and `PiTensorProductPhase.exists_kappa_product_one_of_piTensorProduct_eq_root_smul`; the finite-cycle phase choice is isolated in `exists_fin_complex_unit_cyclic_coboundary_shift_of_prod_eq_one`. The one-step transport theorem is closed by the single-site overlap theorem above together with the overlap-to-gauge theorem and the proved sector primitivity.[14]

Corner-letter building blocks. The corner-transition tensors $A^i_u = P_u A^i P_{u+1}$ (eq:Auprop) and their repeated products $F^{\mathbf{i}}_u = A^{i_1}_u A^{i_2}_{u+1} \cdots$ (eq:Fu) are now defined as `MPSTensor.cornerLetter` and `MPSTensor.cornerProd`. The cyclic concatenation law $F^{\mathbf{i}_1\mathbf{i}_2}_u = F^{\mathbf{i}_1}_u F^{\mathbf{i}_2}_{u+\ell_1}$, by which the appendix repeats a single block around the cycle to an injective length, is `MPSTensor.cornerProd_append`; the junction projector is absorbed by idempotency, so the law needs no injectivity or normality input. These are the A^i_u/F_u building blocks that enter the first bullet above. The formal right-inverse contraction of F_u to *eq:resultprop* and the unitary global gauge assembly are completed in `sectorTensor_proportional_of_blockedMatch`.

6 Scope restriction: independence without spanning

Source.[15] As a consequence of the proposition and Lemma *Lem1t*, for $N \geq N_0$ the non-zero members of $\{|V_N(A_j)\rangle\}_{j\in J}$ are linearly independent *and* span $|V_N(A)\rangle$; the spanning half is what "justifies the name basis of periodic vectors".

Lean statement.[16] Only the independence half is stated; the spanning clause is dropped. The basis condition is encoded directly as the pairwise non-repetition hypothesis `hNonrep` together with a common-period divisibility `hDiv`, rather than derived from a basis of periodic tensors, and the *Lem1t* almost-orthonormal-implies-independent lemma is not used. Restoring the spanning

[13] Formal declaration: `sectorTensor_proportional_of_blockedMatch` in `SectorMatch/Contraction.lean`.

[14] The formal declarations are `sectorGaugePhaseEquiv_succ_of_cyclicTransport` and `gaugePhaseEquiv_of_overlap_norm_tendsto_one_of_irreducible_TP`.

[15] [1], lines 604–608 and Lemma *Lem1t*, lines 511–519. The phrase about the name "basis of periodic vectors" appears at line 611.

[16] Formal declaration: `periodicBasis_eventuallyLinearlyIndependent` in `Dichotomy.lean`; the matching blueprint entry is in `blueprint/src/chapter/ch22_periodic_ft_overlap_sector_match_and_consequences.tex`.

clause and routing through *Lem1t* was classified under issue #81. The independence half is formalized; no unrestricted spanning assertion is presently attributed to the source consequence.

7 Multiplicity-bearing overlap theorem and Perron transport

Source.[17] The irreducible forms used in the proportional Fundamental Theorem are

$$A^i = \bigoplus_j (R_j \otimes A^i_j), \qquad\qquad B^i = \bigoplus_k (S_k \otimes B^i_k), \tag{4}$$

where the diagonal matrices R_j and S_k retain the multiplicities and weights of repeated copies of each periodic basis tensor.

Multiplicity-bearing normalized theorem. The declaration `PeriodicOverlapHypothesis.ofSectorDecompositions` uses literal sector decompositions

$$P^i = \bigoplus_{j,q} (\mu_{j,q} A^i_j), \qquad\qquad Q^i = \bigoplus_{k,r} (\nu_{k,r} B^i_k), \tag{5}$$

where every complex weight is nonzero. This flattened presentation is the coordinate form of $\bigoplus_j (R_j \otimes A^i_j)$ and $\bigoplus_k (S_k \otimes B^i_k)$, with

$$\operatorname{tr}(R_j^N) = \sum_q \mu^N_{j,q}, \qquad\qquad \operatorname{tr}(S_k^N) = \sum_r \nu^N_{k,r}. \tag{6}$$

It assumes proportionality only at positive lengths, permits different total bond dimensions, and allows the proportionality scalar to depend freely on the length. Thus the multiplicity, literal-representation, and empty-word restrictions of the earlier scalar theorem are absent.

The proof makes the paper's terse contradiction at line 631 explicit. It partitions one basis by mixed-overlap decay, passes to a common-period subsequence, and compares the grouped coefficients in an eventually linearly independent coproduct-indexed family. Linear independence forces $\operatorname{tr}(R^{pN}_{j_0}) = 0$ for every sufficiently large N. Geometric extrapolation applied to the nonzero numbers $\mu^p_{j_0,q}$ then forces the same power sum to vanish at exponent zero, contradicting the positive multiplicity of the sector. In particular, the proof never treats the repeated copies of A_{j_0} as independent vectors.

The earlier declaration `PeriodicOverlapHypothesis.ofIsIrreducibleForm` remains as the one-weight-per-listed-block specialization. Its hypothesis still asserts matrix-product-vector equality at the empty word and may describe

[17] [1], equations *bdnr* and *Bbdnr*, lines 286–305 and 575–585; the normalization reduction, lines 313–332; and theorem *thm:bd*, lines 613–632.

an ambient tensor only through such an equality. These are restrictions of that older formulation, not of the new literal multiplicity-bearing theorem.

Pure normalization theorem. The predicate `IsSpectrallyPeriodic` records the source assumptions on a basis block: irreducibility, spectral radius one, positive period, and the prescribed roots-of-unity peripheral spectrum. The declaration `IsSpectrallyPeriodic.exists_isPeriodic_tpGauge` applies the positive adjoint Perron eigenmatrix to obtain a left-canonical periodic block by pure similarity, exactly as in [1, lines 313–332]. The Perron eigenvalue is forced to equal one, so no scalar rescales the block. The peripheral spectrum is preserved by similarity. Thus this normalization leaves every matrix-product vector unchanged and introduces no extra power into the multiplicity coefficients.

Sectorwise normalization and return to the original blocks. The declaration `SectorDecomposition.replaceBasis` changes only the representative tensors, keeping their index set, bond dimensions, multiplicities, and weights fixed. If X_j is the pure Perron gauge of the jth representative, then `SectorDecomposition.gaugeEquiv_toTensor_replaceBasis` assembles the global gauge

$$X = \bigoplus_j (I_{r_j} \otimes X_j). \tag{7}$$

Consequently each block $\mu_{j,q} A_j^i$ is sent to $\mu_{j,q} X_j A_j^i X_j^{-1}$, with no change to $\mu_{j,q}$. The declaration `SectorDecomposition.exists_isPeriodic_replaceBasis` applies this construction simultaneously to every spectrally periodic representative.

Pure similarities preserve all closed-chain coefficients, so they preserve every mixed overlap exactly. The declaration `PeriodicOverlapHypothesis.of_gaugeEquiv` uses this equality to preserve decay and non-decay, and composes the pure gauges with any repeated-block relation obtained after normalization. Pairwise non-repetition is preserved by the same transitivity argument. Finally, `PeriodicOverlapHypothesis.ofSpectrallyPeriodicSectorDecompositions` normalizes both literal multiplicity-bearing decompositions, transports their positive-length proportionality to the normalized tensors, applies the normalized multiplicity theorem, and returns the complete periodic-overlap hypothesis to the original spectral-radius-one blocks. This completes the overlap-hypothesis step tracked by issue #5342.

The declarations `fundamentalTheorem_periodic_proportional` and `fundamentalTheorem_periodic_proportional_of_isPeriodic` begin after this step: they accept the periodic-overlap hypothesis, or its non-decaying partner fields, as premises and prove the resulting finite block matching. Together with the spectrally periodic multiplicity theorem they give the block matching for literal irreducible forms. The exact source-labelled theorem remains *not ready*, however, because the existing matching conclusion does not explicitly retain equality of the matched periods. This is the remaining gap intrinsic to Theorem 3.4 at lines 613–632. Separately, extending the irreducible-form theorem to arbitrary input tensors through Proposition 2.1 requires the positive-length, possibly lower-dimensional comparison at lines 266–271; that extension is not part of Theorem 3.4.

Remaining elimination plan. Combine the spectrally periodic overlap theorem with the finite matching argument in a source-labelled statement that retains equality of matched periods. For the separate extension beyond Theorem 3.4, prove the positive-length comparison with the representative from Proposition 2.1. The completed overlap-hypothesis step is tracked by issue #5342; the subsequent multiplicity-power extraction is tracked by issue #5344.

References

[1] G. de las Cuevas, J. I. Cirac, N. Schuch, and D. Pérez-García, *Irreducible forms of matrix product states: Theory and applications*, J. Math. Phys. **58**, 121901 (2017), arXiv:1708.00029.

[2] J. I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, *Matrix product states and projected entangled pair states: Concepts, symmetries, theorems*, Ann. Phys. **378**, 100 (2017) ("Ci15" in arXiv:1708.00029).

[3] M. M. Wolf, *Quantum Channels and Operations: Guided Tour*, unpublished lecture notes (2012).