Shifted Trace Powers and the Exact MPU Transfer Spectrum

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Verdict: open-gap (resolved).

1 Source statement

Let U be a matrix-product unitary of physical dimension d and bond dimension D. Combine its two physical indices into $a = (p, q) \in \{1, \ldots, d\}^2$ and define its normalized flattening by $\widetilde{U}^{(p,q)} = d^{-1/2} U^{p,q}$. Its transfer map is the linear endomorphism $T : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ given by

$$T(X) = \sum_{a \in \{1,\ldots,d\}^2} \widetilde{U}^a X (\widetilde{U}^a)^\dagger. \tag{1}$$

This is `MPSTensor.transferMap` applied to `MPOTensor.normalizedFlattening`. For matrix units $F_{k\ell}$, let $E = \widehat{T}$ be the column-stacked transfer-matrix representation of T. Explicitly, column stacking means $\mathrm{vec}(X)_{(j,i)} = X_{ij}$, and E is characterized by

$$E_{(j,i),(\ell,k)} = \big(T(F_{k\ell})\big)_{ij}, \tag{2}$$

$$E\ \mathrm{vec}(X) = \mathrm{vec}(T(X)). \tag{3}$$

These identities are the defining convention of `transferMatrix` and its application lemma `transferMatrix_mulVec_eq`. Thus E is indexed by pairs of bond indices and has matrix dimension $n = D^2$. For any $n \times n$ matrix B, write

$$\chi_B(X) = \det(X\mathbb{1}_n - B) \tag{4}$$

for its characteristic polynomial; in particular, χ_E denotes the characteristic polynomial of the transfer matrix.

For every integer $N > 1$, Proposition III.1 of [CPGSV17] derives the shifted moment identity

$$\mathrm{tr}(E^N) = 1. \tag{shifted moments}$$

It concludes that the sole nonzero eigenvalue of E is 1, with algebraic multiplicity one. Equivalently,

$$\chi_E(X) = X^{n-1}(X-1). \tag{source conclusion}$$

The cited Lemma A.5 of [CPGSV16] is naturally stated with all positive moments, including the first moment. The MPU hypothesis above does not supply $\mathrm{tr}(E) = 1$.

2 Exact shifted-moment argument

Fix the transfer matrix E, its dimension n, and its characteristic polynomial χ_E as defined above. The missing first moment is unnecessary. For every $m \geq 1$, the shifted moments imply

$$\mathrm{tr}((E^2)^m) = \mathrm{tr}(E^{2m}) = 1, \qquad \mathrm{tr}((E^3)^m) = \mathrm{tr}(E^{3m}) = 1. \tag{5}$$

The all-positive-moment characteristic-polynomial theorem therefore gives

$$\chi_{E^2}(X) = \chi_{E^3}(X) = X^{n-1}(X-1). \tag{6}$$

If $\lambda \neq 0$ is a spectral value of E, spectral mapping gives

$$\lambda^2 = 1, \qquad \lambda^3 = 1, \qquad \lambda = 1. \tag{7}$$

Conversely, spectral mapping from $1 \in \sigma(E^2)$ supplies a nonzero spectral value of E, so

$$\sigma(E) \setminus \{0\} = \{1\}. \tag{8}$$

This set equality alone does not determine algebraic multiplicity.

For multiplicity, use the commuting factorization

$$E^2 - \mathbb{1} = (E + \mathbb{1})(E - \mathbb{1}). \tag{9}$$

Every generalized 1-eigenvector of E is consequently a generalized 1-eigenvector of E^2. Hence

$$\mathrm{mult}_1(\chi_E) \leq \mathrm{mult}_1(\chi_{E^2}) = 1. \tag{10}$$

Since $1 \in \sigma(E)$, the left-hand side is positive and therefore equals one. All other nonzero roots have already been excluded. Splitting the monic characteristic polynomial over $\mathbb{C}$ yields

$$\mathrm{mult}_1(\chi_E) = 1, \qquad \chi_E(X) = X^{n-1}(X-1). \tag{11}$$

No identity for $\mathrm{tr}(E)$, positivity, normality, or diagonalizability is used.

3 Verdict and formalization status

Verdict: the shifted-moment multiplicity gap is closed. The set-spectrum statement remains available as a deliberately narrower consequence, but the exact source conclusion is now formalized directly from moments with exponent greater than one.[1] For the normalized flattening of an MPU of bond dimension D, the specialization gives the exponent $D^2 - 1$.[2]

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.

[1] Formal declarations: `Matrix.spectrum_diff_zero_eq_singleton_of_forall_trace_pow_eq_one_of_one_lt`, `Matrix.charpoly_rootMultiplicity_one_eq_one_of_forall_trace_pow_eq_one_of_one_lt`, and `Matrix.charpoly_eq_X_pow_pred_mul_X_sub_one_of_forall_trace_pow_eq_one_of_one_lt` in `TNLean/Algebra/ShiftedTracePowerSpectrum`.

[2] Formal declaration: `MPOTensor.IsMPU.normalizedFlattening_charpoly` in `TNLean/MPS/MPU/TransferMatrix`.