

Corrections to matrix product density operator fixed-point statements

We examine several fixed-point, convergence, and entropy statements in Cirac–Pérez-García–Schuch–Verstraete. The counterexamples concern the simple-MPDO implication chain in Theorem ??, the pure-state zero-correlation-length equivalences, convergence of canonical-form renormalization flows, the thermodynamic limit in Proposition ??, and the finite-ring saturation claim in Example ??. For the map from two to three sites asserted in Proposition ??, an explicit tensor satisfies all standing hypotheses but has $\|K_2(Y)\|_1 = 337/250 < 27/20 = \|K_3(Y)\|_1 = \|K_4(Y)\|_1$, excluding any such trace-preserving completely positive map by trace-norm contraction. We present these counterexamples alongside a coefficient-separation proof for the fundamental theorem of matrix product vectors and several local corrections.

Verdict: false-source (resolved).

I. SCOPE AND CONVENTIONS

The statements examined in Ref. [1] concern three distinct questions: fixed-point characterizations, existence of limits, and exact finite-ring entropy identities. Table I identifies each source claim and its treatment here. Section II examines Proposition ?? and Theorem ??. Section III addresses canonical coefficients and phase cycles in pure and mixed renormalization flows. Section IV discusses entropy limits and finite-ring area-law saturation. Section V provides a proof completion for the fundamental theorem of matrix product vectors, and Sec. VI collects local corrections. Source references use arXiv:1606.00608v4 numbering; unqualified equation references are local.

The verification remarks below identify the counterexamples and corrected statements formalized in Lean in the TNLean library. They refer to the stated conclusions and hypotheses; the proofs in this note are written in ordinary mathematical language. The same-phase convergence argument, the limsup amendment, and the replacement for Example ?? are established here by the accompanying calculations and are not included in the formalization claims below.

For an MPDO tensor K with physical dimension d , bond dimension D , and boundary matrix $X \in M_D(\mathbb{C})$, define

$$K_N(X) = \sum_{\mathbf{i}, \mathbf{j}} \text{tr}(X K^{i_1 j_1} \dots K^{i_N j_N}) |\mathbf{i}\rangle \langle \mathbf{j}|, \quad (1)$$

$$\sigma^{(N)}(K) = K_N(\mathbb{1}_D).$$

Here $M_D(\mathbb{C})$ denotes the $D \times D$ matrices. The tensor K generates translationally invariant operators $\sigma^{(N)}(K) \geq 0$. Mixed-state zero correlation length (ZCL) uses the first of two distinct transfer objects:

$$B_K = \sum_i K^{ii}, \quad \mathcal{E}_K(H) = \sum_{i,j} K^{ij} H (K^{ij})^\dagger, \quad (2)$$

$$K \text{ has mixed ZCL} \iff B_K^2 = B_K. \quad (3)$$

A renormalization fixed point (RFP) requires trace-preserving completely positive maps (tpCPMs):

$$\exists \mathcal{T}, \mathcal{S} \text{ tpCPM} : \begin{cases} \mathcal{T}[K_1(X)] = K_2(X), \\ \mathcal{S}[K_2(X)] = K_1(X) \end{cases} \quad \forall X \in M_D(\mathbb{C}). \quad (4)$$

We use the canonical form (CF), normal tensors, and basis of normal tensors (BNT) as defined in Ref. [1]. For normal representatives $\mathcal{K}_a$, the relevant conditions are

$$\begin{aligned} & \{\mathcal{K}_a^{ij}\}_{i,j} \text{ irreducible}, \quad r(\mathcal{E}_{\mathcal{K}_a}) = 1, \\ & \text{spec}(\mathcal{E}_{\mathcal{K}_a}) \cap \{|z| = 1\} = \{1\}, \\ & \text{biCF} : \quad \text{span}_{i,j} \{\oplus_a \mathcal{K}_a^{ij}\} = \oplus_a M_{D_a}(\mathbb{C}), \\ & 0 < |\mu_{a,q}| \leq 1, \quad \max_{a,q} |\mu_{a,q}| = 1, \\ & \text{simple} : \quad B_{\mathcal{K}_a} = \sum_i \mathcal{K}_a^{ii} \text{ is not nilpotent} \quad \forall a. \end{aligned} \quad (5)$$

TABLE I. Source statements and their treatment. Each claim is read with its hypotheses in Ref. [1]; only the conclusion at issue is abbreviated here. CF denotes canonical form, RFP a renormalization fixed point, ZCL zero correlation length, and SAL saturation of the area law. The notation $K^{[2]}$ means two-site blocking, $\mathcal{E}$ denotes the pure-state transfer map, and $I_L^{(N)}$ is the mutual information of an L -site block in an N -site ring; $\sim$ denotes similarity.

Original statement	Claim at issue	Treatment here
Proposition ??; Theorem ??(ii) $\Rightarrow$ (v) Theorem ??(iv) $\Rightarrow$ (v)	Maps from two to three sites; SAL + ZCL $\Rightarrow K^{[2]}$ is an RFP Structure of normal representatives $\Rightarrow K^{[2]}$ is an RFP	Counterexample, Sec. II Copy-weight counterexample, Sec. III
Theorems ?? and ??	Pure ZCL $\Leftrightarrow \mathcal{E}^2 = \mathcal{E} \Leftrightarrow$ RFP	Weight and adjacency obstructions, Secs. III and VI
Convergence assertion after Theorem ?? and in Appendix B Appendix D, after Eq. (??)	$\mathcal{E}^{2^n}$ converges for every CF initial tensor $\mathcal{E}^2 \sim \mathcal{E} \Rightarrow \mathcal{E}^2 = \mathcal{E}$	Phase-cycle counterexample, Sec. III Gauge-blocking counterexample, Sec. III
Proposition ??	$\lim_{N \rightarrow \infty} I_L^{(N)}$ exists at fixed L	Parity counterexample, Sec. IV
Example ??	The displayed finite-periodic family has SAL: $I_2^{(4)} = I_1^{(4)}$	Strict entropy defect, Sec. IV
Theorem ??; Corollary ??	Sector matching and equal-MPV weight recovery	Proof completion, Sec. V

Here r is the spectral radius, biCF means block-injective canonical form, and $\mu_{a,q}$ are the canonical copy weights. Only active weights are retained. The unit-modulus weight is required globally, not in each sector.

Entropies use base-two logarithms and the normalized periodic state:

$$\hat{\sigma}^{(N)} = \frac{\sigma^{(N)}}{\text{tr } \sigma^{(N)}}, \quad I_L^{(N)} = S_L^{(N)} + S_{N-L}^{(N)} - S_N^{(N)}. \quad (6)$$

Here $S_L^{(N)}$ is the entropy of a connected L -site block. Saturation of the area law (SAL) means $I_L^{(N)} = I_1^{(N)}$ for $1 \leq L \leq \lfloor N/2 \rfloor$. Entropy normalization does not rescale K in a channel identity.

II. LOCAL MAPS FOR SIMPLE TENSORS WITH SATURATION AND ZERO CORRELATION LENGTH

In Appendix C.2 of Ref. [1], the standing setup considers a simple tensor K in block-injective canonical form (biCF) that satisfies saturation of the area law (SAL) and zero correlation length (ZCL). Under these assumptions, Proposition ?? asserts that there exist two trace-preserving completely positive maps (tpCPMs) $\mathcal{T}$ and $\mathcal{S}$ acting on the physical indices,

$$\begin{aligned} \mathcal{T} : M_{d^2}(\mathbb{C}) &\rightarrow M_{d^3}(\mathbb{C}), \\ \mathcal{S} : M_{d^3}(\mathbb{C}) &\rightarrow M_{d^2}(\mathbb{C}), \end{aligned} \quad (7)$$

such that for every boundary matrix $X \in M_D(\mathbb{C})$,

$$\mathcal{T}[K_2(X)] = K_3(X), \quad \mathcal{S}[K_3(X)] = K_2(X). \quad (8)$$

This is the identity represented in diagram (??). Theorem ?? then defines M as the two-site block of K [Eq. (??)] and concludes that M is a renormalization fixed point (RFP) via the implication chain (ii) $\Rightarrow$ (v). By Definition ??, an RFP map for M requires a tpCPM mapping $M_1(X) = K_2(X)$ to $M_2(X) = K_4(X)$ for all $X \in M_D(\mathbb{C})$.

The following construction satisfies all standing hypotheses of Appendix C.2, but directly tests and excludes these maps.

A. The four-letter tensor

For $i = 0, 1, 2, 3$, set

$$\begin{aligned} t &= (1, 2, -7, 4)^\top, & s &= (-3, -1, 1, 3)^\top, \\ l_i &= \begin{pmatrix} 1 \\ t_i \end{pmatrix}, & r_i &= (1/4 \ s_i/100), \\ F^{ii} &= F_i = l_i r_i, & F^{ij} &= 0 \quad (i \neq j). \end{aligned} \tag{9}$$

Thus $d_F = 4$ and $D_F = 2$. With $L = (l_0, \dots, l_3)$ and R having rows r_i ,

$$\sum_i t_i = \sum_i s_i = \sum_i s_i t_i = 0, \quad LR = P = \text{diag}(1, 0), \tag{10}$$

$$T := RL = \frac{1}{100} \begin{pmatrix} 22 & 19 & 46 & 13 \\ 24 & 23 & 32 & 21 \\ 26 & 27 & 18 & 29 \\ 28 & 31 & 4 & 37 \end{pmatrix}. \tag{11}$$

In particular,

$$\begin{aligned} T_{ij} &= \frac{1}{4} + \frac{s_i t_j}{100} > 0, & \mathbf{1} &= (1, 1, 1, 1)^\top, \\ T\mathbf{1} &= \mathbf{1}, & \mathbf{1}^\top T &= \mathbf{1}^\top. \end{aligned} \tag{12}$$

Writing $Q = \mathbf{1}\mathbf{1}^\top/4$ and $N_0 = st^\top/100$ gives

$$\begin{aligned} T &= Q + N_0, & Q^2 &= Q, & QN_0 &= N_0Q = N_0^2 = 0, \\ T^2 &= Q, & T^m &= Q \quad (m \geq 2). \end{aligned} \tag{13}$$

Rank-one multiplication yields

$$\begin{aligned} F_{i_1} \cdots F_{i_N} &= l_{i_1} \left(\prod_{a=1}^{N-1} T_{i_a i_{a+1}} \right) r_{i_N}, \\ \text{tr}(F_{i_1} \cdots F_{i_N}) &= \prod_{a=1}^N T_{i_a i_{a+1}} > 0, & i_{N+1} &= i_1. \end{aligned} \tag{14}$$

Consequently, without rescaling,

$$\begin{aligned} \sigma^{(N)}(F) &\geq 0, & \text{tr } \sigma^{(N)}(F) &= \text{tr } T^N = 1 \quad (N \geq 1), \\ B_F &= LR = P = P^2. \end{aligned} \tag{15}$$

B. Exact finite-ring SAL

The periodic distribution and its proper connected marginals are

$$\begin{aligned} p_N(i_1, \dots, i_N) &= \prod_{a=1}^N T_{i_a i_{a+1}}, \\ p_{N,L}(i_1, \dots, i_L) &= \left(\prod_{a=1}^{L-1} T_{i_a i_{a+1}} \right) (T^{N-L+1})_{i_L i_1} \\ &= \frac{1}{4} \prod_{a=1}^{L-1} T_{i_a i_{a+1}}, & 1 \leq L < N. \end{aligned} \tag{16}$$

The complementary exponent $N - L + 1 \geq 2$ makes Eq. (13) applicable even when one site is traced out. Define

$$h = -\frac{1}{4} \sum_{i,j} T_{ij} \log_2 T_{ij}. \quad (17)$$

The initial letter is uniform and every internal edge has probability $T_{ij}/4$, so

$$\begin{aligned} S_L^{(N)} &= 2 + (L-1)h \quad (1 \leq L < N), \\ p_N(i_a = i, i_{a+1} = j) &= T_{ij}(T^{N-1})_{ji} = T_{ij}/4 \quad (N \geq 3), \\ S_N^{(N)} &= Nh \quad (N \geq 3), \\ I_L^{(N)} &= 4 - 2h \quad (N \geq 3, 1 \leq L < N). \end{aligned} \quad (18)$$

This proves finite-ring SAL; at $N = 2$ there is only one nonempty cut size.

C. Normal form and the global canonical weight

The coordinate matrix of the four slices, ordered as (11, 12, 21, 22), satisfies

$$\begin{aligned} \det \left[\frac{1}{4}, \frac{s_i}{100}, \frac{t_i}{4}, \frac{s_i t_i}{100} \right]_{i=0}^3 &= -\frac{3}{2000} \neq 0, \\ \text{span}\{F_i\} &= M_2(\mathbb{C}). \end{aligned} \quad (19)$$

Hence $\mathcal{E}_F$ is strictly positive:

$$v^\dagger \mathcal{E}_F(H)v = \sum_i \|H^{1/2} F_i^\dagger v\|^2 > 0 \quad (H \geq 0, H \neq 0, v \neq 0). \quad (20)$$

Equality would force $H^{1/2} A^\dagger v = 0$ for every $A \in M_2(\mathbb{C})$, whereas $\{A^\dagger v : A \in M_2(\mathbb{C})\} = \mathbb{C}^2$. Perron–Frobenius theory therefore gives positive left and right eigenmatrices, unique up to positive scalar multiples:

$$\mathcal{E}_F(\rho) = r\rho, \quad \mathcal{E}_F^*(H_*) = rH_*, \quad \rho, H_* > 0, \quad r > 0, \quad (21)$$

with no other peripheral eigenvalue; $\mathcal{E}_F^*$ is the trace adjoint.

For $H = \text{diag}(20, 1)$,

$$\begin{aligned} H - \mathcal{E}_F^*(H) &= \begin{pmatrix} 85/8 & -9/40 \\ -9/40 & 4697/5000 \end{pmatrix} > 0, \\ \det(H - \mathcal{E}_F^*(H)) &= \frac{19861}{2000} > 0. \end{aligned} \quad (22)$$

Pairing with ρ shows

$$0 < \text{tr}[(H - \mathcal{E}_F^*(H))\rho] = (1 - r) \text{tr}(H\rho), \quad 0 < r < 1. \quad (23)$$

Put $\mu = \sqrt{r}$, $G = H_*^{1/2}$, and $\mathcal{K}_0^{ii} = \mu^{-1} G F_i G^{-1}$. Then

$$\sum_i (\mathcal{K}_0^{ii})^\dagger \mathcal{K}_0^{ii} = r^{-1} G^{-1} \mathcal{E}_F^*(H_*) G^{-1} = \mathbb{1}_2, \quad (24)$$

so $\mathcal{K}_0$ is normal.

Embed its physical support in the first four coordinates of $\mathbb{C}^5$, and take the bond-one normal tensor $\mathcal{K}_1^{44} = 1$ on the fifth coordinate, with other slices zero. The required ambient tensor is

$$K = (\mu \mathcal{K}_0) \oplus \mathcal{K}_1, \quad d_K = 5, \quad D_K = 3. \quad (25)$$

Its simultaneous canonical conditions are

$$\begin{aligned}
\text{span}_{i,j}\{\mathcal{K}_0^{ij} \oplus \mathcal{K}_1^{ij}\} &= M_2(\mathbb{C}) \oplus \mathbb{C}, \\
(\mu_{0,1}, \mu_{1,1}) &= (\mu, 1), \quad 0 < \mu < 1 = \max_{j,q} |\mu_{j,q}|, \\
B_{\mathcal{K}_0} &= \mu^{-1} G P G^{-1}, \quad B_{\mathcal{K}_1} = 1, \\
B_{\mathcal{K}_j}^n &\neq 0 \quad (j = 0, 1, n \geq 1).
\end{aligned} \tag{26}$$

Orthogonal physical supports make these representatives a BNT; the span and nonnilpotence prove biCF and simplicity. For the coordinate inclusion $V : \mathbb{C}^4 \rightarrow \mathbb{C}^5$,

$$\begin{aligned}
\sigma^{(N)}(K) &= V^{\otimes N} \sigma^{(N)}(F) (V^{\otimes N})^\dagger + |4^N\rangle \langle 4^N| \geq 0, \\
\text{tr } \sigma^{(N)}(K) &= 2 \quad (N \geq 1), \\
S_L^{(N)}(K) &= 1 + \frac{1}{2} S_L^{(N)}(F) \quad (1 \leq L \leq N), \\
I_L^{(N)}(K) &= 1 + \frac{1}{2} I_L^{(N)}(F) \quad (1 \leq L < N), \\
B_K &= (G P G^{-1}) \oplus 1, \quad B_K^2 = B_K.
\end{aligned} \tag{27}$$

Thus positivity of generated states, global weight normalization, biCF, simplicity, finite-ring SAL, and literal ZCL hold together.

D. A boundary excluding trace-preserving completely positive maps

Choose

$$X = \begin{pmatrix} -1 & 2/5 \\ -5 & 0 \end{pmatrix}. \tag{28}$$

The endpoint coefficients in Eq. (14) are

$$\begin{aligned}
D_{ij} &= r_j X l_i = -\frac{1}{4} + \frac{t_i}{10} - \frac{s_j}{20}, \\
D &= \begin{pmatrix} 0 & -1/10 & -1/5 & -3/10 \\ 1/10 & 0 & -1/10 & -1/5 \\ -4/5 & -9/10 & -1 & -11/10 \\ 3/10 & 1/5 & 1/10 & 0 \end{pmatrix}.
\end{aligned} \tag{29}$$

Although X need not be Hermitian, every closure is real diagonal:

$$\begin{aligned}
F_N(X) &= \sum_{\mathbf{i}} D_{i_1 i_N} \prod_{a=1}^{N-1} T_{i_a i_{a+1}} |\mathbf{i}\rangle \langle \mathbf{i}| = F_N(X)^\dagger, \\
\|F_N(X)\|_1 &= \sum_{i,j} |D_{ij}| (T^{N-1})_{ij}.
\end{aligned} \tag{30}$$

The displayed matrices give

$$\begin{aligned}
\|F_2(X)\|_1 &= \frac{337}{250} < \frac{27}{20} = \|F_3(X)\|_1 = \|F_4(X)\|_1, \\
\frac{27}{20} - \frac{337}{250} &= \frac{1}{500} > 0.
\end{aligned} \tag{31}$$

Use the same ambient boundary at every length:

$$\begin{aligned}
Y &= (G X G^{-1}) \oplus 0, \\
K_N(Y) &= V^{\otimes N} F_N(X) (V^{\otimes N})^\dagger.
\end{aligned} \tag{32}$$

Cyclicity cancels the gauges, the zero boundary removes the fifth-letter sector, and the isometry preserves the trace norm.

Every positive trace-preserving map contracts that norm on Hermitian inputs. For $Z = Z_+ - Z_-$,

$$\|\Phi(Z)\|_1 \leq \text{tr } \Phi(Z_+) + \text{tr } \Phi(Z_-) = \text{tr } Z_+ + \text{tr } Z_- = \|Z\|_1. \quad (33)$$

Thus the failed conclusions are

$$\nexists \Phi \text{ tpCPM} : \quad \Phi[K_2(Y)] = K_m(Y), \quad m = 3, 4. \quad (34)$$

The $m = 3$ inequality refutes the all-boundary identity of Proposition ???. If M denotes the two-site block of K ,

$$M_1(Y) = K_2(Y), \quad M_2(Y) = K_4(Y), \quad M \text{ is not an RFP.} \quad (35)$$

Refutation. For the boundary Y , Eq. (31) contradicts the trace-norm contraction required by Eq. (8). The same contradiction at four sites excludes the RFP map for the two-site block in Theorem ??(ii) $\Rightarrow$ (v). Both conclusions therefore fail under the printed standing hypotheses.

A corrected implication would require additional hypotheses; the results here do not determine such a criterion. The complete counterexample, including all standing hypotheses and the nonexistence of both tpCPMs, is formalized in Lean.¹

E. Two points in the structural proof

The matrix T also exhibits the missing Jordan-block issue in source Lemma ??:

$$\begin{aligned} T &= Q + N_0 > 0, \quad N_0 \neq 0, \quad N_0^2 = 0, \\ \text{spec}(T) &= \{0, 1\}, \quad \text{tr } T^n = 1 \quad (n \geq 1), \\ T_{00}T_{11} - T_{01}T_{10} &= \frac{22 \cdot 23 - 19 \cdot 24}{10000} = \frac{1}{200} \neq 0. \end{aligned} \quad (36)$$

Refutation. Here $T > 0$ means entrywise positivity. Lemma ?? asserts that if T is primitive with $\text{tr}(T^N) = \text{tr } T = 1$ for all N , then T has rank one, $T_{k,h} = a_k b_h$. The matrix $T = Q + N_0$ in Eq. (36) satisfies those trace conditions, yet has rank two because $T_{00}T_{11} - T_{01}T_{10} = 1/200 \neq 0$.

For an injective normal tensor with SAL and literal ZCL, the rank-one neighboring-trace factorization is established because the normal self-overlap limit isolates a single active sector. This corrected Case-I statement is formalized in Lean under injectivity, normality, SAL, and the stated ZCL identity.²

In Case II, the argument absorbs the canonical copy weights μ_j into the normal representatives $\mathcal{K}_j$:

$$\mathcal{E}_{\mu_j \mathcal{K}_j} = |\mu_j|^2 \mathcal{E}_{\mathcal{K}_j}, \quad r(\mathcal{E}_{\mu_j \mathcal{K}_j}) = |\mu_j|^2. \quad (37)$$

Refutation. Equation (37) shows that this absorption scales the transfer spectral radius to $|\mu_j|^2$. When $|\mu_j| < 1$, as for $\mu \mathcal{K}_0$ in Eq. (25), the absorbed representative loses normality, so this repair does not extend to general simple tensors.

III. CANONICAL COEFFICIENTS AND RELATIVE PHASES

A. Physical ZCL does not determine the canonical coefficients

For pure tensors, the source uses correlations independent of distance (CID) and local orthogonality (LO):

$$\begin{aligned} \text{pure ZCL} &\iff (\text{CID and LO}), \\ \text{LO} : \sum_i A_j^i \otimes \overline{A_{j'}^i} &= 0 \quad (j \neq j'). \end{aligned} \quad (38)$$

¹ Lean proof of the Proposition C.15 and blocked-RFP counterexample.

² Lean construction of the active-sector rank-one factorization.

This differs from mixed ZCL, $B_K^2 = B_K$. The pure transfer map and its canonical copy blocks are

$$\begin{aligned}\mathcal{E}_A(H) &= \sum_i A^i H (A^i)^\dagger, & \mathbb{E}_A &= \sum_i A^i \otimes \overline{A^i}, \\ (\mathbb{E}_A)_{j,q;j',q'} &= \mu_{j,q} \overline{\mu_{j',q'}} \sum_i A_j^i \otimes \overline{A_{j'}^i},\end{aligned}\tag{39}$$

in canonical coordinates. Literal idempotence retains the copy factors.

Take one physical letter:

$$\begin{aligned}A^0 &= \text{diag}(1, 1/2), & A_1^0 &= 1, \\ (\mu_{1,1}, \mu_{1,2}) &= (1, 1/2), \\ |V^{(N)}(A)\rangle &= (1 + 2^{-N})|0^N\rangle.\end{aligned}\tag{40}$$

This tensor is in CF; LO is vacuous. All physical observables are scalar, so for observables O_1, O_2 on disjoint regions and every translation τ_Δ allowed by CID,

$$\langle V^{(N)} | O_1 O_2 | V^{(N)} \rangle = \langle V^{(N)} | O_1 \tau_\Delta(O_2) | V^{(N)} \rangle.\tag{41}$$

CID holds even for unnormalized expectations at fixed N , but for the matrix unit e_{12} ,

$$\mathcal{E}_A(e_{12}) = \frac{1}{2}e_{12}, \quad \mathcal{E}_A^2(e_{12}) = \frac{1}{4}e_{12}.\tag{42}$$

Refutation. Thus pure ZCL does not imply idempotence in source Theorem ??, or the unrestricted equivalence of Theorem ??.

Directly, the RFP equation gives

$$(A^0)^2 = u A^0 \implies u = 1, \quad \frac{1}{4} = \frac{1}{2},\tag{43}$$

a contradiction. Theorems ?? and ?? require that pure ZCL be equivalent to transfer idempotence $\mathbb{E}^2 = \mathbb{E}$ and to the RFP property. An established replacement restricts the assertion to the basis direct sum containing exactly one unit-weight copy of each distinct BNT representative. For that direct sum, positive-gap physical CID combined with BNT local orthogonality is equivalent to transfer idempotence $\mathbb{E}^2 = \mathbb{E}$. Replacing the original tensor by this basis direct sum modifies the selected tensor and can change its normalized physical family. Both the counterexample with coefficients $(1, 1/2)$ and this unit-coefficient representative theorem are formalized in Lean.³

As an MPDO tensor, $K^{00} = \text{diag}(1, 1/2)$ independently tests Theorem ??(iv) $\Rightarrow$ (v):

$$\begin{aligned}\mathcal{K}_1^{00} &= 1, & \text{span}\{\mathcal{K}_1^{00}\} &= \mathbb{C}, & B_{\mathcal{K}_1} &= 1, \\ \max_q |\mu_{1,q}| &= 1, & \sigma^{(N)}(K) &= 1 + 2^{-N} > 0, \\ \eta &= 1, & a = b &= 1.\end{aligned}\tag{44}$$

These satisfy simplicity, biCF, global normalization, positivity, and the complete scalar factorization in condition (iv). Trace preservation for its two-site block requires

$$(K^{00})^4 = (K^{00})^2, \quad \text{diag}(1, 1/16) = \text{diag}(1, 1/4),\tag{45}$$

Refutation. The required identity fails. Condition (iv) constrains the normal representatives without restricting the canonical coefficients $\mu_{j,q}$, and copy-independent weights alone are insufficient when weights are strictly below one.

A sufficient strengthened theorem requires exact assembly up to gauge, simultaneous one-site span, and copy-independent weights. The weight-absorbed representatives must satisfy layer orthogonality and admit physical-sector factorizations with positive neighboring operators obeying $\text{tr} \eta_{kh} = a_k b_h$ and $\sum_k a_k b_k = 1$. Specializing all copy weights to $\mu_{j,q} = 1$ while retaining biCF, layer orthogonality, and the physical-sector factorizations of condition (iv) suffices to prove that the blocked tensor is an RFP. The counterexample and the sufficient theorem with the stated factorization hypotheses have Lean proofs.⁴

³ Canonical-coefficient counterexample; positive-gap ZCL equivalence for the unit-coefficient basis direct sum.

⁴ Counterexample to Theorem 4.9(iv) to (v); sufficient theorem for coefficient-absorbed factorizations.

B. Peripheral phases and similarity under squaring

Let

$$\omega = e^{2\pi i/3}, \quad A^0 = \text{diag}(1, \omega), \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (46)$$

Both scalar normal copies have unit-modulus weights, yet

$$\mathcal{E}_A^{2^n}(e_{21}) = \omega^{2^n} e_{21} = \begin{cases} \omega e_{21}, & n \text{ even}, \\ \omega^2 e_{21}, & n \text{ odd}. \end{cases} \quad (47)$$

Refutation. This contradicts the universal CF convergence assertion following source Theorem ?? and in Appendix B.

A sufficient condition for convergence of $\mathcal{E}^{2^n}$ is that all unit-modulus copies within each normal representative share the same phase. Under this condition, surviving relative phase factors equal one, subunit factors decay, and cross terms between distinct normal representatives decay.

The construction also tests the pure-state equivalence asserted after source Eq. (??):

$$\begin{aligned} \omega S(A^0)^2 S^{-1} &= \omega \text{diag}(\omega^2, 1) = A^0, \\ M &= A^0 \otimes \overline{A^0}, \\ M &= (S \otimes \overline{S}) M^2 (S \otimes \overline{S})^{-1}, \quad M^2 \neq M. \end{aligned} \quad (48)$$

The doubled tensor is in CF, with scalar normal copies and

$$(\mu_q)_q = (1, \overline{\omega}, \omega, 1), \quad \rho^{(N)}(M) = |1 + \omega^N|^2 > 0. \quad (49)$$

Refutation. Identity maps on the one-dimensional physical algebras satisfy both virtual-gauge equations in Appendix D. Equation (48) also gives $M^2 \neq M$, which violates the ordinary RFP condition. Thus the asserted equivalence is false.

Virtual conjugation preserves an idempotent transfer map; obtaining idempotence from the blocking-up-to-gauge condition requires a further hypothesis excluding the peripheral cycles. No replacement equivalence is proved here. Lean proofs verify the failure of dyadic convergence and the blocking-up-to-gauge counterexample, including canonical form and normalization of the doubled tensor.⁵

Theorem ?? remains valid as a limit characterization:

$$\begin{aligned} \mathcal{E}^{2^n} \longrightarrow F &\implies F^2 = \lim_n \mathcal{E}^{2^{n+1}} = F, \\ F^2 = F &\implies F^{2^n} = F. \end{aligned} \quad (50)$$

The correspondence between completely positive maps and physical isometry classes of Kraus tensors yields the stated characterization whenever a limit exists.

IV. FINITE-CHAIN ENTROPY AND THERMODYNAMIC LIMITS

A. The inner thermodynamic limit

Consider the CF tensor

$$\begin{aligned} M^{00} &= \text{diag}(1, 0, 0), & M^{11} &= \text{diag}(0, 1, -1), \\ M^{01} &= 0, & M^{10} &= 0. \end{aligned} \quad (51)$$

⁵ Nonconvergence of the dyadic transfer powers; pure blocking-up-to-gauge counterexample.

Its scalar normal copies have globally normalized weights $1, 1, -1$. Mixed words vanish, giving

$$\begin{aligned}\rho^{(N)} &= |0^N\rangle\langle 0^N| + (1 + (-1)^N)|1^N\rangle\langle 1^N| \geq 0, \\ \text{tr } \rho^{(N)} &= 2 + (-1)^N.\end{aligned}\tag{52}$$

At odd length, the normalized state and all nonempty marginals are pure; at even length, their nonzero eigenvalues are $1/3, 2/3$. With $h_2(p) = -p \log_2 p - (1-p) \log_2 (1-p)$,

$$I_L^{(N)} = \begin{cases} 0, & N \text{ odd}, \\ h_2(1/3), & N \text{ even} \end{cases} \quad (N > L \geq 1).\tag{53}$$

Therefore

$$\liminf_{N \rightarrow \infty} I_L^{(N)} = 0 < h_2(1/3) = \limsup_{N \rightarrow \infty} I_L^{(N)} \quad (L \geq 1),\tag{54}$$

Refutation. The the unrestricted inner limit in source Proposition ?? does not exist.

The MPDO property and failure of this limit at every fixed positive cut are formalized in Lean.⁶

The finite-chain bounds remain separate. Strong subadditivity on consecutive lengths $1, L, N - 2L - 1$, with translation invariance, yields

$$\begin{aligned}S_L^{(N)} + S_{N-L}^{(N)} &\leq S_{L+1}^{(N)} + S_{N-L-1}^{(N)}, \\ I_L^{(N)} &\leq I_{L+1}^{(N)} \quad (1 \leq L < \lfloor N/2 \rfloor).\end{aligned}\tag{55}$$

The source proof also quotes $I_L^{(N)} \leq 4 \log_2 D$. These bounds establish outer convergence whenever each fixed-cut inner limit exists:

$$\begin{aligned}J_L := \lim_{N \rightarrow \infty} I_L^{(N)} &\implies J_L \leq J_{L+1} \leq 4 \log_2 D, \\ \lim_{L \rightarrow \infty} J_L &\leq 4 \log_2 D.\end{aligned}\tag{56}$$

Replacing the inner limit by $\limsup_{N \rightarrow \infty} I_L^{(N)}$ gives a bounded monotone sequence in L , so its outer limit exists. The original iterated-limit formulation requires hypotheses ensuring the fixed-cut limits; such hypotheses are not derived here.

B. The literal periodic state in Example ??

In source Example ??, site n is in $|k_n, k_n\rangle$. The nonzero spectrum is the periodic distribution

$$\begin{aligned}p_N(k_1, \dots, k_N) &= \frac{1}{Z_N} \prod_{n=1}^N W_{k_n k_{n+1}}, \quad k_{N+1} = k_1, \\ W &= \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.\end{aligned}\tag{57}$$

With $P_{\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$,

$$\begin{aligned}W^m &= (3/2)^m P_+ + (1/2)^m P_-, \\ Z_N &= \text{tr } W^N = (3^N + 1)/2^N, \\ p_{N,L}(k_1, \dots, k_L) &= \frac{\prod_{n=1}^{L-1} W_{k_n k_{n+1}} (W^{N-L+1})_{k_L k_1}}{Z_N} \\ &\quad (1 \leq L < N).\end{aligned}\tag{58}$$

⁶ Lean proof of the Proposition 4.5 limit counterexample.

The complementary path is essential on a finite ring.

At $N = 4$,

$$\begin{aligned} Z_4 &= \frac{41}{8}, & W^2 &= \begin{pmatrix} 5/4 & 1 \\ 1 & 5/4 \end{pmatrix}, \\ W^3 &= \begin{pmatrix} 7/4 & 13/8 \\ 13/8 & 7/4 \end{pmatrix}. \end{aligned} \quad (59)$$

The exact marginal probabilities are

$$\begin{aligned} p_{4,1}(0) &= p_{4,1}(1) = \frac{1}{2}, \\ p_{4,2}(00) &= p_{4,2}(11) = \frac{14}{41}, \\ p_{4,2}(01) &= p_{4,2}(10) = \frac{13}{82}, \end{aligned} \quad (60)$$

$$\begin{aligned} p_{4,3}(k_1, k_2, k_3) &= \begin{cases} 10/41, & 0 \text{ internal transitions (2 words)}, \\ 4/41, & 1 \text{ internal transition (4 words)}, \\ 5/82, & 2 \text{ internal transitions (2 words)}. \end{cases} \end{aligned} \quad (61)$$

Their entropies give

$$\begin{aligned} I_2 - I_1 &= 2S_2 - S_1 - S_3 \\ &= \frac{1}{82} \log_2 \frac{41^{82} 5^{50}}{2^{48} 13^{52} 7^{112}} > 0. \end{aligned} \quad (62)$$

Strictness has a short conditional-MI proof. Translation invariance identifies the defect with $I(k_1 : k_3 \mid k_2)$, and

$$\begin{aligned} q_{ab} &:= \Pr(k_1 = a, k_3 = b \mid k_2 = 0) \\ &= \frac{1}{41} \begin{pmatrix} 20 & 8 \\ 8 & 5 \end{pmatrix}_{ab}, \quad \det q = \frac{100 - 64}{41^2} > 0. \end{aligned} \quad (63)$$

Thus q is not the product p of its marginals. For positive probability distributions,

$$\begin{aligned} \sum_a q_a \ln(q_a/p_a) &\geq \sum_a (q_a - p_a) = 0, \\ &\text{equality iff } q = p, \end{aligned} \quad (64)$$

Refutation. by $-\ln x \geq 1 - x$. The conditional mutual information is strictly positive, so the literal four-ring fails SAL.

The non-ZCL claim remains correct:

$$p_{4,2}(00) = \frac{14}{41} \neq \frac{5}{14} = \frac{(W^2)_{00}}{\text{tr } W^3} = p_{3,2}(00), \quad 196 \neq 205. \quad (65)$$

Literal physical-trace idempotence would identify these reduced closures and their normalizations. Lean proofs establish both the failure of SAL and non-idempotence of the physical-trace matrix for the original two-qubit-site tensor.⁷ A replacement for the failed SAL example is an MPDO with diagonal physical slices $K^{00} = \text{diag}(1, 0)$ and $K^{11} = \text{diag}(0, c)$ for a fixed constant $0 < c < 1$, with all off-diagonal slices zero. This generates the family $\sigma^{(N)} = |0^N\rangle\langle 0^N| + c^N |1^N\rangle\langle 1^N|$. For every proper connected cut $1 \leq L < N$, tracing $N - L$ spins leaves the two nonzero eigenvalues $1/(1 + c^N)$ and $c^N/(1 + c^N)$, matching the normalized spectrum of the full state. Consequently $S_L^{(N)} = S_N^{(N)} = h_2(c^N/(1 + c^N))$ and $I_L^{(N)} = h_2(c^N/(1 + c^N))$ for every proper cut, giving exact SAL across cuts on every finite ring. The physical-trace matrix $B_K = \text{diag}(1, c)$ has $B_K^2 = \text{diag}(1, c^2) \neq B_K$, so ZCL fails. This supplies the intended SAL-without-ZCL example.

⁷ Failure of SAL for the ambient Example 4.11 tensor; non-idempotence of its physical-trace matrix.

V. PROOF COMPLETION FOR THE FUNDAMENTAL THEOREM

This section supplies two steps omitted from the published argument: coefficient comparison in a relation containing both complete BNT families, and recovery of the canonical coefficients from identities valid only at sufficiently large lengths. These steps establish the intended conclusions of Theorem ?? and Corollary ??.

Theorem ?? states that if two tensors A and B in CF generate proportional MPV families $|V^{(N)}(A)\rangle = c_N |V^{(N)}(B)\rangle$, with $c_N \neq 0$ for every positive N , then their BNTs have the same cardinality $g_a = g_b$ and match up to phases and similarity transformations. Corollary ?? concludes that if the families are equal ($c_N = 1$), their matrices are related by a global similarity $A^i = XB^iX^{-1}$. The source proof compares one B -sector with the A -family using Lemma ??, but the proportionality relation contains all B -sectors simultaneously. Full-family coefficient comparison supplies the missing step. Retain only active, nonzero canonical weights; zero-weight blocks carry no dimension information.

For distinct nonzero $z_1, \dots, z_m$,

$$\begin{aligned} \sum_{a=1}^m c_a z_a^N &= 0 \quad (N \geq N_0), \\ \det[z_a^{N_0+b}]_{\substack{0 \leq b < m \\ 1 \leq a \leq m}} &= \left(\prod_{a=1}^m z_a^{N_0} \right) \prod_{a < b} (z_b - z_a) \neq 0 \\ \implies c_a &= 0 \quad \forall a. \end{aligned} \tag{66}$$

After equal weights are grouped, a nonempty copy power sum has positive integer multiplicities. It therefore cannot vanish on an entire tail.

Form the union of both BNT families, quotienting first by

$$\begin{aligned} C \sim D &\iff \exists \phi \in \mathbb{R}, X \text{ invertible} : \\ D^i &= e^{i\phi} X C^i X^{-1} \quad \forall i. \end{aligned} \tag{67}$$

Choose one representative C_u per class. The source normal-tensor overlap lemma gives

$$\begin{aligned} G_{uv}^{(N)} &:= \langle V^{(N)}(C_u) | V^{(N)}(C_v) \rangle \longrightarrow \delta_{uv}, \\ \det G^{(N)} &\neq 0 \quad (N \geq N_0). \end{aligned} \tag{68}$$

Hence the entire quotient family is jointly linearly independent on a tail.

Absorb the matching phases into the coefficients and write

$$\begin{aligned} |V^{(N)}(A)\rangle &= \sum_u a_u(N) |V^{(N)}(C_u)\rangle, \\ |V^{(N)}(B)\rangle &= \sum_u b_u(N) |V^{(N)}(C_u)\rangle. \end{aligned} \tag{69}$$

Then the proportionality hypothesis gives the full-family comparison

$$\begin{aligned} |V^{(N)}(A)\rangle &= c_N |V^{(N)}(B)\rangle, \quad c_N \neq 0 \\ \implies a_u(N) &= c_N b_u(N) \quad \forall u, N \geq N_0. \end{aligned} \tag{70}$$

Choosing an unmatched A -sector as its class representative would give

$$b_u(N) = 0, \quad a_u(N) = \sum_q \mu_{j,q}^N = 0 \quad (N \geq N_0), \tag{71}$$

contradicting the determinant argument. Interchanging the families excludes unmatched B -sectors. Thus phase-gauge matching is bijective, without a unit-weight assumption in each sector.

For equal MPVs, choose matched representatives $B_j^i = e^{i\phi_j} Y_j A_j^i Y_j^{-1}$. Coefficient comparison gives

$$\begin{aligned} \sum_q \mu_{j,q}^N &= \sum_q (e^{i\phi_j} \nu_{j,q})^N \quad (N \geq N_0), \\ \{\mu_{j,q}\}_q &= \{e^{i\phi_j} \nu_{j,q}\}_q \quad \text{as multisets.} \end{aligned} \tag{72}$$

The second line follows by grouping distinct nonzero weights and applying the determinant argument to their multiplicity differences. The sums then agree at every positive length; matching copies gives the global similarity in Corollary ???. For proportional families, the conclusion establishes sector matching. The proportional sector-matching theorem and the equal-family global-similarity theorem are formalized in Lean, including the passage from eventual coefficient identities to equality of coefficient multisets.⁸

VI. LOCAL CORRECTIONS

A. CID requires positive complementary gaps

The forward implication from transfer idempotence to printed CID has an independent adjacency obstruction. Take

$$A^{(a,b)} = \frac{1}{\sqrt{2}}|a\rangle\langle b|, \quad a, b \in \{0, 1\}. \quad (73)$$

Then

$$\begin{aligned} \text{span}\{A^{(a,b)}\} &= M_2(\mathbb{C}), \\ \mathcal{E}_A(H) &= \tfrac{1}{2} \text{tr}(H) \mathbb{1}_2, \quad \mathcal{E}_A^2 = \mathcal{E}_A. \end{aligned} \quad (74)$$

This is a single normal RFP with no copy weights. Its periodic MPS is a product of Bell pairs between the right qubit of each site and the left qubit of the next.

On a four-ring, let $Z_{n,r}$ and $Z_{n,l}$ denote Pauli Z on the respective qubits of site n . With $|\varphi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$,

$$\begin{aligned} \langle Z_{1,r} Z_{2,l} \rangle &= \langle \varphi | Z \otimes Z | \varphi \rangle = 1, \\ \langle Z_{1,r} Z_{3,l} \rangle &= \left[\tfrac{1}{2}(1 - 1) \right]^2 = 0. \end{aligned} \quad (75)$$

Refutation. The printed CID interval permits this translation because the regions remain disjoint. Thus idempotence does not imply CID for adjacent regions.

Let g_1, g_2 and g'_1, g'_2 be the complementary gap lengths before and after translation. The corrected forward statement requires

$$\begin{aligned} g_1, g_2, g'_1, g'_2 &\geq 1, \quad \mathbb{E}^2 = \mathbb{E} \\ \implies \mathbb{E}^{g_1} &= \mathbb{E}^{g_2} = \mathbb{E}^{g'_1} = \mathbb{E}^{g'_2} = \mathbb{E}. \end{aligned} \quad (76)$$

At adjacency, $\mathbb{E}^0 = \mathbb{1}$ instead. The resulting statement is an adjacency correction for spatial correlators. The Bell-pair counterexample and the positive-gap implication are both formalized in Lean.⁹

B. Square-root normalization

In source Theorem ?? and Lemma ??, the pair-index isometry condition implies

$$\sum_i U_{ab}^i \overline{U_{a'b'}^i} = \delta_{aa'} \delta_{bb'} \implies \sum_i U^i H (U^i)^\dagger = \mathbb{1} \text{tr } H. \quad (77)$$

For a trace-normalized diagonal density fixed point,

$$\begin{aligned} \Lambda > 0, \quad \text{tr } \Lambda &= 1, \\ \begin{cases} A^i = \sqrt{\Lambda} U^i & \Rightarrow \mathcal{E}_A(H) = \Lambda \text{tr } H, \\ A^i = \Lambda U^i & \Rightarrow \mathcal{E}_A(H) = \Lambda^2 \text{tr } H. \end{cases} \end{aligned} \quad (78)$$

⁸ Equal-family global similarity; proportional sector matching; coefficient-multiset recovery from eventual identities.

⁹ Bell-pair adjacency counterexample; positive-gap CID from transfer idempotence.

The printed second form has spectral radius $\text{tr}(\Lambda^2)$; for bond dimension $D > 1$, positivity and $\text{tr} \Lambda = 1$ give $\text{tr}(\Lambda^2) < 1$. Including the gauge, the corrected forms are equivalently

$$\begin{aligned} A^i &= X \sqrt{\Lambda} U^i X^{-1}, \quad \text{tr} \Lambda = 1, \\ \text{or} \quad A^i &= X \Lambda U^i X^{-1}, \quad \text{tr}(\Lambda^2) = 1. \end{aligned} \quad (79)$$

In the first convention, the bond state has amplitudes $\sqrt{\Lambda_{aa}}$; in the second, Λ is the Schmidt-amplitude matrix. Both conventions yield normal RFP tensors.

C. Indices, exponents, and endpoints

Source Eq. (??) must leave i_1, i_2 free and sum only over j :

$$A^{i_1} A^{i_2} = \sum_j U_{(i_1, i_2), j} A^j. \quad (80)$$

In “Commuting parent Hamiltonians”, the length- L support vectors are

$$|v_{\alpha, \beta}\rangle = \sum_{i_1, \dots, i_L} (A^{i_1} \cdots A^{i_L})_{\alpha, \beta} |i_1, \dots, i_L\rangle. \quad (81)$$

In the notation of Appendix C’s proof of Theorem ??,

$$\begin{aligned} A_\gamma &= \frac{m_\gamma}{n_\gamma} U_\gamma M_\gamma U_\gamma^\dagger \\ \implies O_L(A_\gamma) &= \left(\frac{m_\gamma}{n_\gamma} \right)^L O_L(M_\gamma). \end{aligned} \quad (82)$$

Here O_L is the vertically contracted length- L operator, and $m_\gamma = \text{tr} \mu_\gamma$, $n_\gamma = \text{tr} \nu_\gamma$ are the traces of the one-site and blocked multiplicity matrices. Thus the first simultaneous blocked-operator representation has coefficient

$$\left(\frac{m_\gamma}{n_\gamma} \right)^L \text{tr}(\nu_\gamma^L), \quad \text{not} \quad \frac{m_\gamma^L}{n_\gamma} \text{tr}(\nu_\gamma^L), \quad (83)$$

as in the later trace-power expression of that proof.

The entropy equality following SAL is

$$I_L = I_{L+1} \iff S_L + S_{N-L} = S_{L+1} + S_{N-L-1}. \quad (84)$$

The closing proof of Theorem ?? cyclically interchanges conditions (iii) and (iv) in its proposition assignments; Proposition ?? starts from condition (ii).

Finally, source Example ?? requires

$$U_n = \sigma_x^{(n, l)} \otimes \sigma_x^{(n, r)}, \quad \mathcal{E}_n(X) = p U_n X U_n^\dagger + (1 - p) X. \quad (85)$$

The exception list for strict failure of SAL must include $p = 1$:

$$p = 1 \implies S_L = 2 \ (1 \leq L < N), \quad S_N = 0, \quad I_L = 4. \quad (86)$$

Deterministic on-site unitary conjugation preserves the Bell-pair ring entropies, with two maximally mixed boundary qubits at every proper connected cut. The non-SAL witness at $p = 1/4$ is unaffected.

[1] J. I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, [Annals of Physics](#) **378**, 100 (2017).