Removing Length-Zero Bookkeeping from the Blocked Canonical-Form Decomposition

2026-05-30

Verdict: clarification (resolved).
This note explains why carrying the blocked canonical-form decomposition as an *all-length* matrix-product-vector identity is the wrong formal shape, and records its removal from the physical canonical-form statements.[1]

1 The assertion

Let A be a translation-invariant matrix-product tensor of bond dimension D, with one-site matrices $A^i \in M_D(\mathbb{C})$. For a length N and a word $\sigma = (i_1, \ldots, i_N)$ write the matrix-product-vector coefficient

$$V^{(N)}(A)_\sigma = \operatorname{tr}\big(A^{i_1} \cdots A^{i_N}\big)\,. \tag{1}$$

The blocked canonical-form reduction of [CPGSV16, Section 2.3], following [PGVWC07, Theorem 3], states that after blocking by some period $p \geq 1$ the tensor is, up to a gauge, the direct sum of an all-zero block and a weighted family of normal blocks,

$$A^{[p]} \cong \mathbf{0}_z \;\oplus\; \bigoplus_k \mu_k\, B_k, \tag{2}$$

where $\mathbf{0}_z$ is the zero tensor on a bond space of dimension z, the B_k are normal (primitive, trace-preserving, irreducible) blocks of bond dimension D_k, the weights satisfy $\mu_k \neq 0$, and $z + \sum_k D_k = D$ is the bond dimension of $A^{[p]}$.

2 The point at issue

The direct sum (2) gives, at every length N and word σ,

$$V^{(N)}\Big(A^{[p]}\Big)_\sigma = V^{(N)}(\mathbf{0}_z)_\sigma + V^{(N)}(\textstyle\bigoplus_k \mu_k B_k)_\sigma\,. \tag{3}$$

[1] Context: PRs PR #2153, PR #2156, and PR #2167 removed roughly a thousand lines of "zero-tail" plumbing built on the all-length form discussed here. This note records why that form is a non-source-faithful overcomplication and why it has now been removed.

The zero block contributes nothing except at the empty word: for $N \geq 1$ a product of A^i-blocks that passes through the zero summand vanishes, so

$$V^{(N)}(\mathbf{0}_z)_\sigma = 0 \qquad (N \geq 1), \tag{4}$$

whereas at $N = 0$ the only coefficient is the trace of the identity on the bond space,

$$V^{(0)}(C)_\varnothing = \mathrm{tr}(\mathbf{1}_{D_C}) = D_C. \tag{5}$$

Hence the *entire* content of the length-zero instance of (3) is the dimension count

$$z + \sum_k D_k = D, \tag{6}$$

which is already part of (2). Every length $N \geq 1$ carries the genuine mathematical content: the weighted normal-block comparison $V^{(N)}\big(A^{[p]}\big)_\sigma = V^{(N)}(\bigoplus_k \mu_k B_k)_\sigma$.

Why the all-length form is the wrong shape. Stating the reduction as the all-length identity (3) attaches the trivial dimension fact (6) to the substantive positive-length comparison and forces both to travel together through blocking, reindexing, and sector grouping. In the formal development this produced a family of transport lemmas whose only purpose was to move the empty-word instance of (3) across each such step.[2] This is also not source-faithful: the article introduces the zero block once, to account for the bond dimension, and then discards it, comparing only the nonzero normal blocks in the thermodynamic limit. The zero summand $\mathbf{0}_z$ is an artifact of keeping a common bond space in the formal statement; it carries no state and no normal-block structure.

3 The clean statement

Separate the two facts. The positive-length comparison

$$V^{(N)}\Big(A^{[p]}\Big)_\sigma = V^{(N)}(\bigoplus_k \mu_k B_k)_\sigma \qquad (N \geq 1) \tag{7}$$

carries all of the physical mathematics. The direct-sum induction independently gives the structural identity for the unfiltered irreducible blocks. After the all-zero blocks are discarded, this implies only the bound needed downstream,

$$\sum_k D_k \leq D. \tag{8}$$

Thus no physical theorem needs an empty-word coefficient or a scalar recording the discarded bond-space dimension. Formally, positive-length equality is `Same MPVPos`; the canonical-form reduction carries (7) and (8).

[2] These were the `zeroTail_*` lemmas in `TNLean/MPS/CanonicalForm/SectorComparison/`; they are removed in PRs PR #2156 and PR #2167 once the comparison is stated at positive length.

4 Formal status

`exists_tp_primitive_blockDecomp_after_blocking`, blueprint `thm:tp_primitive_blockdecomp` in `blueprint/src/chapter/ch09_canonical.tex`, now uses the source-faithful shape: its MPV clause is the positive-length comparison in (7), and its dimension clause is the bound (8). The intermediate declarations carrying `zeroTailDim`, as well as the former support lemma `zeroTail_toTensorFromBlocks_blockPower`, have therefore been removed. The subsequent common-sector theorems now also take `SameMPVPos` directly; no all-length hypothesis is converted back to positive length inside their proofs. The former bridge from positive-length equality plus equal bond dimensions to an all-length statement has been removed from the public comparison surface. The empty-word coefficient lemma remains only as an algebraic identity in the Lean source and is not presented as a physical chain statement in the blueprint.

5 Conclusion

The block decomposition (2) is correct as an intermediate linear-algebra statement. The former formal defect was one of shape: stating the reduction as the all-length matrix-product-vector identity (3) fused a trivial empty-word dimension count with the substantive positive-length comparison. The physical statement is the positive-length comparison (7) together with the structural bound (8). The development now uses this shape throughout the after-blocking primitive decomposition and its downstream consumers.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[PGVWC07] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7(5–6):401–430, 2007.