Zero correlation length uses the physical-trace transfer in Cirac–Perez-Garcia–Schuch–Verstraete 2017

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Verdict: open-gap (open).

1 Source statement

For mixed states, Definition 4.2 (local label `DefinitionZCL`, `Papers/1606.00608/MPDO-22-12-17-2.tex:735--739`) defines a tensor M generating an MPDO to have *zero correlation length* (ZCL) by a graphical equation displayed as the figure `MPDO_ZCL.png`. The diagram closes the ket and bra physical legs of a single MPDO tensor and compares the resulting one-layer virtual transfer with its square. In formulas, if M^{ij} denotes the bond matrix with ket index i and bra index j, the object represented by the diagram is the physical-trace transfer

$$\mathcal{T}_M := \sum_i M^{ii},$$

not the doubled-index completely positive map on bond matrices. The text states that this is "the natural extension of the pure state case of Theorem **??**" and that correlation functions computed from an MPDO with ZCL are length-independent.

The pure-state case is made precise in the proof of Theorem **??** (`Papers/1606.00608/MPDO-22-12-17-2.tex:1248`): ZCL "is equivalent to the property $\mathbb{E}^2 = \mathbb{E}$ for a tensor A in CF" — that is, the transfer operator is idempotent *for a tensor in canonical form.* The renormalization step is $\mathcal{E} \mapsto \mathcal{E}^2$ (`Papers/1606.00608/MPDO-22-12-17-2.tex:1209`), and a fixed point satisfies $\mathcal{E} = \mathcal{E}^2$. Canonical form normalizes the transfer operator so that its leading eigenvalue equals 1; the idempotence $\mathbb{E}^2 = \mathbb{E}$ is the statement that, after this normalization, $\mathbb{E}$ projects onto the peripheral (eigenvalue-1) subspace.

2 The gap

The Lean definition `MPOTensor.IsZCL` instead reads

$$\mathcal{E}_M \circ \mathcal{E}_M = \mathcal{E}_M, \qquad \mathcal{E}_M(X) = \sum_{i,j} M^{ij} X (M^{ij})^*,$$

i.e. *literal* idempotence of the doubled-index CP transfer map $\mathcal{E}_M$. This is not the transfer drawn in Definition 4.2: the source contracts the two physical legs first and squares $\mathcal{T}_M = \sum_i M^{ii}$. Thus the current predicate is not merely an un-normalized version of the source condition; it is a condition on a different transfer object. Even after allowing a scalar normalization, idempotence of $\mathcal{E}_M$ would classify tensors by the doubled-index MPS transfer rather than by the MPDO ZCL diagram.

Classification: definition drift. The formal predicate records doubled-index CP-transfer idempotence, while the source definition uses the physical-trace transfer. The latter is now formalized as `MPOTensor.physTraceTransfer`. The predicate `MPOTensor.IsSourceZCL` records this development's scale-invariant up-to-scalar repair; it is distinct from the paper's literal diagram and from the doubled-index predicate.

This deviation is witnessed in the development by `MPOTensor.exists_isLocalPurificationRFP_not_isZCL`. Take $d = d_K = 2$, $D' = D = 1$, $A = [\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}]$, which is an RFP MPS tensor ($E_A = \mathrm{id}$). The induced MPDO tensor has scalar entries $M^{00} = M^{11} = \frac{1}{2}$, off-diagonal 0, so $\rho^{(N)}(M) = (\frac{1}{2})^N I$ is the (normalized) maximally mixed state — manifestly length-independent correlations, hence ZCL in the source's sense. On the source transfer,

$$\mathcal{T}_M = M^{00} + M^{11} = 1,$$

so the graphical idempotence holds. On the Lean doubled-index CP transfer, $\mathcal{E}_M = \frac{1}{2}\,\mathrm{id}$, so $\mathcal{E}_M \circ \mathcal{E}_M = \frac{1}{4}\,\mathrm{id} \neq \mathcal{E}_M$ and literal `IsZCL` fails. The witness therefore exposes the wrong transfer object, not only a missing scalar normalization.

3 Consequence and elimination plan

The source's PRFP $\Rightarrow$ ZCL implication (`Papers/1606.00608/MPDO-22-12-17-2.tex:777`) is therefore not provable against `IsZCL` as literally stated: the source implication controls $\mathcal{T}_M$, whereas the Lean predicate controls $\mathcal{E}_M$. Two possible repairs are:

1. **Source ZCL transfer.** Introduce the physical-trace transfer $\mathcal{T}_M = \sum_i M^{ii}$ and state ZCL as idempotence of this transfer, with scalar normalization allowed at the level of the generated state family: $\mathcal{T}_M \neq 0$ and $\mathcal{T}_M^2 = \lambda \mathcal{T}_M$ for some $\lambda > 0$. The physical-trace-normalized representative is the case $\lambda = 1$.

2. **Canonical physical-trace hypothesis.** Keep exact idempotence $\mathcal{T}_M^2 = \mathcal{T}_M$ only for source-labelled theorems whose hypotheses include the required canonical normalization. The doubled-index map $\mathcal{E}_M$ remains a separate CP transfer used in the current formal fusion and algebra statements, not the source's ZCL transfer.

The scale-invariant repair described in Option 1 is now in place. The predicate `IsZCL` remains documented as a doubled-index CP-transfer idempotence condition, while `IsSourceZCL` records the up-to-scalar physical-trace interpretation. It must not be identified with the paper's literal diagram for a fixed tensor. For the renormalization condition of Definition 4.1, Proposition `propsimple` is now proved at the explicit normalization boundary: every positive-length ring is positive semidefinite and has nonzero trace. The primary source-facing declarations are `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS` for literal Definition 4.2 ZCL and `MPOTensor.isSAL_of_isRFPViaTS_of_trace_ne_zero` for SAL. The conjunction `MPOTensor.isSourceZCL_and_isSAL_of_isRFPViaTS_of_trace_ne_zero` is a convenience theorem with the broader scale-invariant conclusion; it is not the paper-facing statement. The horizontal source-facing specialization uses `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS` together with `MPOTensor.isSAL_of_isRFPViaTS`. The corresponding `IsSourceZCL` conjunction is again convenience-only. Positivity by itself admits the zero tensor, so the trace condition cannot be omitted. The PRFP equivalence remains open. In Theorem 4.9, the literal implication (iv)$\Rightarrow$(v) is now refuted by the unequal repeated-copy example described below, while the source (ii)$\Rightarrow$(v) channel route remains a missing statement after the ZCL-derived common-weight reduction.

Amendment for the fixed-tensor implication in Theorem 4.9. Option 1 is scale-invariant at the level of the generated state family, but it cannot be substituted unchanged into Theorem 4.9's implication to the exact trace-preserving maps of Definition 4.1. The four-sector audit in `docs/paper-gaps/cpgsv17_pf_rank_one.tex` exhibits the distinction. Its raw tensor $\mathcal{K}$ satisfies the literal diagram $\mathcal{T}_{\mathcal{K}}^2 = \mathcal{T}_{\mathcal{K}}$ and has explicit renormalization maps, but its singleton BNT coefficient is strictly subunit. The globally unit-weight representative $A = (4/\sqrt{5})\mathcal{K}$ satisfies `IsSourceZCL` only in the up-to-scalar sense:

$$\mathcal{T}_A^2 = \frac{4}{\sqrt{5}}\mathcal{T}_A \neq \mathcal{T}_A.$$

Its two-site blocking consequently fails Definition 4.1. The scalar absorption at source line 1660 is performed only after literal ZCL has been assumed and does not transport literal ZCL or the renormalization maps across the rescaling. Thus Option 1 is the scale-invariant interpretation used by TNLean, whereas the paper's fixed representative still carries a separate literal diagrammatic condition. Neither of these four-sector representatives settles the exact (ii)$\Rightarrow$(v) route. A separate scalar BNT presentation with raw copy weights 1 and 1/2 does satisfy the global normalization and every clause of condition (iv), while its two-site blocking fails Definition 4.1. This kernel-checked counterexample is

MPOTensor.CaseIIAbsorptionCounterexample.printed_theorem49_iv_to_v_is_false and refutes the literal (iv)⇒(v).

4 The scale-invariant repaired interpretation and its consequences

The repaired interpretation used in this development is Option 1. Writing $\mathcal{T}_M$ for the physical-trace transfer, its condition is

$$\texttt{IsSourceZCL}\ M \ := \ \mathcal{T}_M \neq 0 \ \wedge \ \exists\, \lambda \in \mathbb{R},\ 0 < \lambda \ \wedge \ \mathcal{T}_M^2 = \lambda\, \mathcal{T}_M.$$

The clause $\mathcal{T}_M \neq 0$ excludes the degenerate zero tensor, which otherwise satisfies $\mathcal{T}_M^2 = \lambda \mathcal{T}_M$ vacuously for every λ. No spectral-radius apparatus is required: if $\mathcal{T}_M^2 = \lambda \mathcal{T}_M$ with $\lambda > 0$ and $\mathcal{T}_M \neq 0$, the minimal polynomial of $\mathcal{T}_M$ divides $x(x-\lambda)$, so its eigenvalues lie in $\{0, \lambda\}$ and λ is the leading eigenvalue. Literal idempotence $\mathcal{T}_M^2 = \mathcal{T}_M$ is exactly the $\lambda = 1$ case, the physical-trace-normalized representative. The maximally mixed witness has $\mathcal{T}_M = 1$ and is therefore classified as ZCL by the source transfer, whereas the current doubled-index condition wrongly excludes it.

The correction is to ZCL, not to the doubled-index ZCL predicate. The doubled-index transfer map $\mathcal{E}_M$ remains the object used by the current Lean predicate MPOTensor.IsZCL and by the transfer-map formulation of the fusion/algebra statements. It should no longer be identified with the source ZCL diagram. Literal idempotence of the physical-trace transfer $\mathcal{T}_M^2 = \mathcal{T}_M$ implies normalized ZCL only together with the nonzero hypothesis $\mathcal{T}_M \neq 0$ (or an equivalent generated-MPDO nondegeneracy assumption); without that clause, the zero transfer is idempotent but not ZCL under the definition above. Every source-facing description that calls the doubled-index literal notion "definitionally zero correlation length" must be weakened to a separate bridge, with the nonzero hypothesis stated explicitly.

Definition 4.1 now gives the forward physical-trace equation. The theorem MPOTensor.physTraceTransfer_sq_of_isRFPViaTS formalizes the zero-correlation-length calculation in Appendix C, Proposition propsimple, source lines 1333–1340. This proposition proves the first implication of Theorem 4.9 by establishing both ZCL and SAL. The physical-trace theorem formalizes its ZCL calculation. At the explicit density-family boundary, MPOTensor.isSourceZCL_and_isSAL_of_isRFPViaTS_of_trace_ne_zero combines it with positive-length nonvanishing and MPOTensor.isSAL_of_isRFPViaTS_of_trace_ne_zero. The horizontal declarations are corollaries obtained by deriving that nonvanishing. For every virtual matrix X, trace preservation of the one-to-two map gives

$$\operatorname{tr}(\mathcal{T}_M^2 X) = \operatorname{tr}(M_2(X)) = \operatorname{tr}(M_1(X)) = \operatorname{tr}(\mathcal{T}_M X).$$

Nondegeneracy of the trace pairing yields $\mathcal{T}_M^2 = \mathcal{T}_M$. Together with $\mathcal{T}_M \neq 0$, this gives `MPOTensor.IsSourceZCL`. The explicit nonzero hypothesis is necessary because the bare predicate `IsRFPViaTS` omits the source's standing canonical-form and normalized MPDO assumptions.

Effect on the purification witness. The diagonal purification of Section 3 has $\mathcal{T}_M = 1$, hence satisfies the source ZCL equation already. The earlier reading, that this witness fails zero correlation length, was an artifact of using $\mathcal{E}_M = \frac{1}{2}\,\mathrm{id}$ in place of the physical-trace transfer. Under the scale-invariant repaired interpretation it is a positive example showing that the up-to-scalar condition captures the maximally mixed product state. This does not identify that condition with the paper's literal fixed-representative diagram.

Historical resolution of the doubled-index coercion. Commit `43fb8749` introduced the former blocked-RFP construction, whose proof treated doubled-index idempotence as the zero-correlation-length clause of Theorem 4.9. Commit `74fb83dd` marked that dependence as unfaithful. Commit `32482c47` then removed the coercion and the paper-facing theorem chain, and commit `fc135934` removed the remaining obsolete module. No such coercion route survives in the current development.

The present formalization separates the three transfer notions. The literal Definition 4.2 conclusion $\mathcal{T}_M^2 = \mathcal{T}_M$ follows directly from Definition 4.1 in `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS`. The broader scale-invariant relation is recorded separately by `MPOTensor.IsSourceZCL`, while `MPOTensor.IsZCL` concerns a different doubled-index transfer and is not used as the source's ZCL clause. The doubled-index results in `FusionIsometries.lean`, `AlgebraStructure.lean`, and `CommutingFormBridge.lean` survive only as separately scoped conditional helpers. None is a relocated proof of Theorem 4.9. The normal Case-I structural conclusion is proved by `MPOTensor.exists_physicalSectorFactorization_rank_one_coefficients_of_isSAL_of_literal_ZCL`. Given one such factorization, `MPOTensor.PhysicalSectorFactorization.NeighboringTraceFactorization.blockTwo_isRFPViaTS` composes the two physical channels of Proposition C.7 twice, as observed at source lines 1821–1825, and transports them back to the original physical coordinates.

The remaining obligations belong to the stronger route from condition (ii). First, the Case-II argument still needs a replacement for the false claim that coefficient absorption preserves normality; this is the *missing statement* in the (ii)⇒(iv) node and is analyzed in `docs/paper-gaps/cpgsv17_pf_rank_one.tex`. Second, the earlier Case-II argument at lines 1646–1665 uses literal ZCL to make repeated-copy weights independent of the copy label and absorb their common value. After that normalization, the Proposition C.7 channel pairs must still be combined, using the mutually orthogonal outer physical supports, into a single pair of trace-preserving completely positive maps. This is the *missing statement* in the (ii)⇒(v) node.

By contrast, the raw implication (iv)⇒(v) is not a remaining assembly problem: it is a refuted *unfaithful statement*, and treating the one-factorization result as though outer-sector assembly would prove it is *proof-path drift*. The hidden use of ZCL in the proof of `prop3to4`, through `lemmus`, is a separate instance of the same proof-path drift. The former doubled-index theorem has been removed rather than renamed, and none of these surviving gaps is a doubled-index bridge or a change of terminology.