CPSV16 Vertical-Sector Invertibility and Sector Relabelling

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Verdict: scope-restriction (resolved).

1 The source argument

In the proof of the general mixed-state renormalization fixed-point theorem, CPSV16 introduces the finite-dimensional C^*-algebras

$$\mathcal{A}_1 = \bigoplus_\alpha M_{d^1_\alpha}(\mathbb{C}), \qquad \mathcal{A}_2 = \bigoplus_\beta M_{d^2_\beta}(\mathbb{C})$$

and the transported maps

$$\widetilde{T} = \widetilde{R}_2 T R_1, \qquad \widetilde{S} = \widetilde{R}_1 S R_2.$$

For every horizontal bond matrix X, the source proves

$$\widetilde{T}\left(\bigoplus_\alpha m_\alpha M_\alpha(X)\right) = \bigoplus_\beta n_\beta A_\beta(X), \qquad \widetilde{S}\left(\bigoplus_\beta n_\beta A_\beta(X)\right) = \bigoplus_\alpha m_\alpha M_\alpha(X).$$

Thus the one-site contraction family belongs to $\mathrm{Fix}(\widetilde{S}\widetilde{T})$, and the two-site family belongs to $\mathrm{Fix}(\widetilde{T}\widetilde{S})$ [CPGSV16, Appendix C.4, lines 1972–1980].

The next step is not a linear-spanning argument. CPSV16 invokes the density-weighted fixed-point form of Wolf's Theorem 6.14 and writes

$$\mathcal{F} = U\left(0_r \oplus \bigoplus_k \rho_k \otimes M_{h_k}(\mathbb{C})\right)U^*.$$

If $C_L(V)$ denotes the span of length-L products of the contraction family, then the basis-of-normal-tensors hypothesis gives $C_L(V) = \mathcal{A}_1$ for some $L > 0$. On the other hand,

$$C_L(\mathcal{F}) = U\left(0_r \oplus \bigoplus_k \rho_k^L \otimes M_{h_k}(\mathbb{C})\right)U^*.$$

Every ρ_k is positive definite, so $\dim C_L(\mathcal{F}) = \dim \mathcal{F}$. The inclusions

$$\mathcal{A}_1 = C_L(V) \subseteq C_L(\mathcal{F}) \subseteq \mathcal{A}_1$$

therefore force $\mathcal{F} = \mathcal{A}_1$. This proves $\widetilde{S}\widetilde{T} = \mathrm{id}_{\mathcal{A}_1}$; the other composition is treated symmetrically [CPGSV16, Appendix C.4, lines 1980–1995].

2 The present formal boundary

Three source-specific ingredients preceding the fixed-point classification are proved:

1. the two transported composites fix their respective weighted bond contraction families;
2. for a basis of normal tensors with nonzero weights, length-L products of these contractions span the full sector algebra for some $L > 0$;
3. on the source-generated contraction families, the final compression preserves the active image and hence preserves total trace.

The third item alone does not establish trace preservation on the whole sector algebra. The contraction family itself need not span the algebra linearly; only its length-L products are known to span. Since a linear map fixing $V(X)$ need not fix products $V(X_1) \cdots V(X_L)$, neither trace preservation nor the identity conclusion extends from the generators by linearity. Whole- sector trace preservation is nevertheless proved by the independent maximal-support argument in the next section. The identity conclusion still requires the fixed-point classification.

Two finite-dimensional final implications are formalized. First, if a linear endomorphism fixes every length-L contraction product and these products span, then it is the identity.[1] Its fixed-product hypothesis is not a hypothesis of CPSV16.

Second, let $\mathcal{F}$ be the fixed-point space of the endomorphism and let $C_L(\mathcal{F})$ be the span of its length-L products. If the distinguished contractions themselves are fixed, their length-L products span the ambient sector algebra, and

$$\dim C_L(\mathcal{F}) \leq \dim \mathcal{F},$$

then the endomorphism is the identity.[2] This second implication does not assume that products of distinguished contractions are fixed. It isolates the source's dimension argument. The displayed inequality is now formalized for a positive trace-preserving map on a finite direct sum whenever its trace adjoint

[1] Formal declaration: `MPOTensor.eq_id_of_weightedVerticalBondContractionProducts_fixed` in `TNLean/MPS/MPDO/VerticalSectorGeneration`.

[2] Formal declarations: `MPOTensor.fixedPointProductSpan` and `MPOTensor.eq_id_of_weightedVerticalBondContractions_fixed_of_productSpan_finrank_le` in `TNLean/MPS/MPDO/VerticalSectorGeneration`.

satisfies the Schwarz inequality and it has a positive-definite fixed family.[3] No tensor hypothesis, multiplicativity assumption, or fixedness of products enters this result. It is nevertheless a full-support corollary: unlike Wolf's unrestricted Theorem 6.14, its statement assumes the positive-definite fixed family. The tensor-attached argument must first obtain that family from the fixed contractions and their positive-length product span, as recorded in the elimination plan below.

The positive trace-preserving mean-ergodic retraction on a full matrix algebra and its positive unital trace-adjoint retraction onto the adjoint fixed-point space are formalized. Their construction uses no bounded-orbit hypothesis on the adjoint map. On full support, the adjoint retraction is now identified with the weighted partial-trace block form of Wolf's Proposition 1.5. The trace-adjoint calculation giving the Schrödinger-picture density blocks and the support reduction producing the zero summand of Wolf's Theorem 6.14 are now assembled. Their construction is recorded in the QICLean paper-gap note `https://sirui-lu.com/QICLean/paper-gaps/wolf_theorem6_14_fixed_point_projection_gap.pdf`. For the transported sector maps one must also prove the channel hypotheses on the full direct sums, rather than only on the contraction families. In the retained-coordinate construction, compression by the target coisometry is trace preserving precisely when the precompression output lies in its active image. The established image identity is restricted to the source-generated contractions.

The passage from one full matrix algebra to the dependent direct sums is also complete for this dimension estimate. One compresses to the diagonal blocks, applies the direct-sum map, and embeds the result block diagonally. Positivity, total-trace preservation, and the Schwarz inequality pass to this extension; its fixed points compress injectively to the fixed points of the direct-sum map, while products of direct-sum fixed points embed injectively among the products of fixed points of the extension. The density-block calculation on the extension therefore gives $\dim C_L(\mathcal{F}) \leq \dim \mathcal{F}$ on the original direct sum. Positivity and trace preservation for the two square composites are resolved below. The Schwarz hypothesis of the abstract identity criterion is also resolved. The canonical full-matrix lifts of the two transported maps factor into normalized state preparation, retained-coordinate reindexings, the physical completely positive maps, single-operator conjugations, and controlled partial trace. Consequently the two transported maps and their square composites are completely positive. Since the square composites preserve trace, their trace adjoints are unital completely positive maps and satisfy the Schwarz inequality in every simple summand.[4] Alternatively, the maximal-support witness $T_\infty(\mathbf{1})$ is now available for an arbitrary positive trace-preserving map. The construction of the sup-

[3] Formal declaration: `MPOTensor.fixedPointProductSpan_finrank_le_of_densityBlocks` in `TNLean/MPS/MPDO/VerticalSectorDensityBlocks`.

[4] Formal declarations: `MPOTensor.transportedVerticalSectorT_isKrausDirectSumMap`, `MPOTensor.transportedVerticalSectorS_isKrausDirectSumMap`, and `MPOTensor.transportedVerticalSector_composites_traceAdjointSchwarz` in `TNLean/MPS/MPDO/VerticalSectorCompletePositivity`.

ported endomorphism and its complementary zero summand is now completed in the full-matrix theorem recorded in the same QICLean paper-gap note; it removes the need to assume a positive-definite fixed point before applying the full-matrix classification.

3 Resolution of the composite trace hypothesis

The range inclusions

$$T(R_1(\mathcal{A}_1)) \subseteq R_2(\mathcal{A}_2), \qquad S(R_2(\mathcal{A}_2)) \subseteq R_1(\mathcal{A}_1)$$

are not consequences presently extracted from lines 1974–1980. They are not needed for trace preservation: first the two square composites preserve trace on their full algebras, and then a trace sandwich gives the same conclusion for each individual transported map.

Let F be either square composite. It is positive and trace nonincreasing. Consequently all forward orbits are bounded: a positive orbit stays positive with decreasing trace, and Hermitian decomposition gives the general case. The mean-ergodic image $\rho = P(\mathbf{1})$ is then a positive fixed point whose support contains every fixed point of F. In particular it contains every distinguished vertical bond contraction. Since positive-length products of these contractions span the whole sector algebra, their common support must contain the identity. Hence ρ is positive definite.

The full-matrix statement of this intermediate conclusion is now formalized.[5] A positive trace-nonincreasing endomorphism with fixed factors for which the identity lies in the span of the products of one positive length has a positive-definite fixed point. No product is required to be fixed. The finite-product passage and its combination with the density-block dimension bound are now also formalized.[6] A positive trace-preserving endomorphism of a finite product therefore has a positive-definite fixed family under the same product-span hypothesis; if its trace adjoint satisfies the Schwarz inequality, the fixed contractions and the density-block bound force the endomorphism to be the identity. For the transported composites, complete positivity of the canonical lifts and the established trace preservation now give the required trace-adjoint Schwarz inequalities from the source tensor hypotheses. Instantiating the criterion twice with the fixed contraction families and their positive-length product spans gives

$$\widetilde{S}\widetilde{T} = \mathrm{id}_{\mathcal{A}_1}, \qquad \widetilde{T}\widetilde{S} = \mathrm{id}_{\mathcal{A}_2}\,.$$

[5]Formal declaration: `IsPositiveMap.exists_posDef_fixedPoint_of_traceNonincreasing_of_fixed_product_span` in `TNLean/Channel/FixedPoint/TraceNonincreasingProductSpan`.

[6]Formal declarations: `Matrix.IsPositiveDirectSumMap.exists_posDef_fixedFamily_of_fixed_product_span` in `TNLean/Channel/FixedPoint/TraceNonincreasingDirectSum`, and `MPOTensor.eq_id_of_weightedVerticalBondContractions_fixed_of_traceAdjointSchwarz` in `TNLean/MPS/MPDO/VerticalSectorInvertibility`.

These are precisely the two identity compositions in [CPGSV16, Appendix C.4, lines 1974–1995].[7]

Let F^* be the trace adjoint and define the trace-loss operator

$$\Delta = \mathbf{1} - F^*(\mathbf{1}).$$

Trace nonincrease gives $\mathrm{tr}(\Delta X) = \mathrm{tr}(X) - \mathrm{tr}(F(X)) \geq 0$ for every positive X, and positivity of F makes Δ Hermitian. Thus $\Delta \geq 0$. Since ρ is fixed, $\mathrm{tr}(\rho\Delta) = 0$; positive definiteness of ρ forces $\Delta = 0$, which is equivalent to trace preservation on the full algebra.

Now let $X \geq 0$ belong to the one-site sector algebra. Positivity and trace nonincrease give

$$\mathrm{tr}((\widetilde{S}\widetilde{T})(X)) \leq \mathrm{tr}(\widetilde{T}(X)) \leq \mathrm{tr}(X).$$

The first and last terms are equal by the composite result, hence both inequalities are equalities. Thus $\widetilde{T}$ preserves trace on the positive cone, and therefore on the full algebra by linearity. The other square composite gives the same conclusion for $\widetilde{S}$.

This argument is formalized first for full matrix algebras, then for finite direct sums by block-diagonal extension, and finally for the two transported vertical-sector composites. It assumes neither multiplicativity nor fixedness of products. The only product statement used is the basis-of-normal-tensors conclusion that products of the fixed contraction *factors* span the ambient sector algebra. Thus the trace-preservation subproblem is closed for the individual maps and their square composites even though the active-image inclusions remain unproved.

4 Sector relabelling from the star-algebra equivalence

After proving mutual invertibility, CPSV16 cites the argument of Wolf–Pérez-García, Theorem 8 [CPGSV16, Appendix C.4, lines 1995–2003]. The cited theorem concerns a trace-preserving Schwarz map. Its proof constructs a positive, trace-preserving inverse on the peripheral space and uses the extreme points of the density-operator sections

$$\bigoplus_k M_{d_k}(\mathbb{C}) \otimes \rho_k.$$

The map and its positive inverse send extreme density operators to extreme density operators. Continuity and bijectivity then give a permutation of the simple summands and equality of the corresponding matrix dimensions. On each matched summand, positivity of both directions gives either unitary conjugation or transposed unitary conjugation; the Schwarz condition excludes the transpose alternative [WPG10, Theorem 8 and its proof].

[7] Formal declaration: `MPOTensor.transportedVerticalSector_composites_eq_id` in `TNLean/MPS/MPDO/VerticalSectorIdentity`.

A linear equivalence between two finite products of matrix spaces does not determine their simple summands. The trace adjoints of a mutually inverse CPTP pair have now been proved to be mutually inverse $*$-algebra homomorphisms, by the Schwarz-defect argument above. The sector relabelling therefore reduces to the following two algebraic conclusions:

1. apply uniqueness of the simple summands to this $*$-algebra equivalence, obtaining the sector permutation and equality of the corresponding matrix dimensions;
2. identify each matched simple-summand isomorphism with unitary conjugation.

The first conclusion is now proved for an arbitrary $*$-algebra isomorphism between two finite products of nonzero full matrix algebras. The block ideals are precisely the atoms in the two-sided ideal lattices, so the induced order isomorphism gives an equivalence σ of the block index sets. Its defining matching relation is

$$E(I_i) = J_{\sigma(i)},$$

where E is the given $*$-algebra isomorphism and I_i, J_j are the source and target block ideals. Its restriction to each paired block is a bijective complex-linear map

$$M_{d_i}(\mathbb{C}) \longrightarrow M_{e_{\sigma(i)}}(\mathbb{C}),$$

and comparison of dimensions gives $d_i = e_{\sigma(i)}$. Applying this result to the trace-adjoint $*$-algebra isomorphism proves both this block-ideal matching relation and the dimension equality for every mutually inverse CPTP pair. The second conclusion is now proved while retaining these particular witnesses. The restriction to a paired block is a $*$-algebra isomorphism; after reindexing by the retained dimension equality, Skolem–Noether gives an invertible implementer P. Preservation of the adjoint forces P^*P to be a positive scalar matrix, and normalization therefore gives a unitary implementer on the actual target block.[8] The trace pairing for this unitary star-algebra isomorphism identifies it with the original forward map, not merely with the adjoint of the inverse. Its inverse uses the same retained relabelling and the adjoints of the same unitaries. Applied to the two transported vertical-sector maps, this gives the two full single-summand formulas for $\widetilde{T}$ and $\widetilde{S}$ stated at line 1997. [9]

[8]Formal declarations: `TwoSidedIdeal.exists_blockEquiv_of_ringEquiv_pi_simple` in `TNLean/Algebra/BlockPermutation`; `Matrix.exists_unitary_conj_of_starAlgEquiv_matrix` and `Matrix.exists_unitary_block_implementers_of_starAlgEquiv_pi_matrix` in `TNLean/Algebra/SkolemNoetherUnitary`; `Matrix.exists_blockEquiv_dim_eq_of_starAlgEquiv_pi_matrix` and `Matrix.exists_blockEquiv_dim_eq_unitary_of_mutual_inverse_kraus_direct_sum_maps` in `TNLean/Channel/FixedPoint/DirectSumBlockPermutation`.

[9]Formal declarations: `Matrix.exists_blockEquiv_dim_eq_unitary_forward_of_mutual_inverse_kraus_direct_sum_maps` in `TNLean/Channel/FixedPoint/DirectSumBlockPermutation`, and `MPOTensor.transportedVerticalSector_exists_unitaryBlockEquiv` in `TNLean/MPS/MPDO/VerticalSectorRelabeling`, and `MPOTensor.transportedVerticalSector_exists_unitaryBlockEquiv_coefficient_eq` in `TNLean/MPS/MPDO/VerticalSectorCoefficientComparison`.

5 Elimination plan

The first two steps of the Appendix C.4 argument are complete:

1. complete positivity of the canonical lifts of the transported maps, and hence the adjoint Schwarz inequality for the two trace-preserving square composites, is established without an all-input active-image statement;

2. the finite-product identity criterion is applied to each square composite, using the fixed contraction families and the proved product-generation theorem, and gives both identity compositions;

The inverse-UCP multiplicative-domain step is also complete: the rectangular trace adjoints form mutually inverse $*$-algebra equivalences. The generic simple-summand correspondence and its internal unitary action are complete as well: they give an equivalence of the sector index sets, identify the actual block ideals carried into one another, prove equality of the paired matrix dimensions, and implement every retained paired component by unitary conjugation. The specialization to both original transported maps, with the same relabelling and the same unitaries, is also complete. Finally, applying the transported refinement map to the weighted contraction family gives

$$m_\alpha V_\alpha \iota_\alpha(M_\alpha(X))V_\alpha^* = n_{\sigma(\alpha)} A_{\sigma(\alpha)}(X).$$

Since the positive multiplicity matrix $\nu_{\sigma(\alpha)}$ has nonzero trace, cancellation proves

$$A_{\sigma(\alpha)}(X) = \frac{m_\alpha}{n_{\sigma(\alpha)}} V_\alpha \iota_\alpha(M_\alpha(X))V_\alpha^*,$$

and evaluation on matrix units gives the tensor-letter identity at line 2008. This comparison concerns the traces of the multiplicity matrices. It does not identify their dimensions, the matrices themselves, or their individual diagonal entries.

6 Classification

Verdict: the Appendix C.4 argument through the coefficient identity at line 2008 is closed. The two transported maps and their square composites are trace preserving under the source tensor hypotheses. Their two compositions are identities, and the rectangular trace adjoints form mutually inverse $*$-algebra equivalences. For any such pair, the simple summands are matched by an equivalence of the index sets, the corresponding block ideals are carried into one another by the star-algebra isomorphism, the paired matrix dimensions are equal, and the induced map on each matched summand is unitary conjugation. The corresponding formulas now hold for $\widetilde{T}$ and $\widetilde{S}$ themselves. Their action on the weighted contraction family gives the trace-ratio coefficient formula and its tensor-letter specialization. The stronger active-image inclusions remain unproved and are not needed for this route. No equality of multiplicities, multiplicity matrices, or individual weights is asserted, and no conclusion after Appendix C.4, line 2008 is included.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[WPG10] Michael M. Wolf and David Pérez-García. The inverse eigenvalue problem for quantum channels. Preprint, 2010.