A Mixed-Prefix Repair of the Two-Site Sector Comparison

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Verdict: local-correction (resolved).

1 Source statement and classification

Appendix C.4 of [CPGSV16] compares the normal tensors in the one-site and two-site vertical canonical forms. After relabelling, the comparison at lines 2053–2056 has the form

$$B_\gamma^v = e^{i\theta_\gamma} Z_\gamma A_\gamma^v Z_\gamma^{-1}, \tag{1}$$

where A_γ is a one-site normal representative and B_γ is its matched two-site representative. Line 2057 invokes Proposition 4.13 to conclude

$$e^{i\theta_\gamma} = 1, \qquad Z_\gamma^\dagger Z_\gamma = \omega_\gamma \mathbf{1}, \qquad \omega_\gamma > 0. \tag{2}$$

Consequently $U_\gamma = \omega_\gamma^{-1/2} Z_\gamma$ is unitary and has the same conjugation action as Z_γ.

Local fix (mixed-prefix alignment). The formal proof requires one small geometric clarification which is not explicit in the source. The two-site sector corner must compress the first two physical sites jointly while leaving the remaining sites as single, unblocked copies of the original tensor. Applying the ordinary reflected-chain argument to the blocked tensor would instead leave a tail made of two-site blocks, which cannot be compared directly with the one-site sector corner. The mixed-prefix reflection lemma supplies the required common unblocked tail. This changes no hypothesis or conclusion of the implication Theorem 4.14(ii)⇒(i); it repairs the proof-level alignment at the invocation of Proposition 4.13.

The phase in (1) is already removed by positivity of two consecutive closed-chain coefficients. The discussion below concerns the remaining metric statement in (2).

2 The two corners

Let M be an MPO tensor which generates matrix product density operators, whose doubled-index tensor is in literal CPSV canonical form, and let a tensor-attached BNT algebra clause be fixed. For one matched label, write A for its one-site normal representative, B for the corresponding two-site representative, and Z for the exact invertible gauge, so that

$$B^v = ZA^vZ^{-1}. \tag{3}$$

The one-site vertical coisometric decomposition has a distinguished positive copy of A. Its inclusion gives a matrix W and a scalar $a > 0$ satisfying

$$W^\dagger W = \mathbf{1}, \qquad W^\dagger \widetilde{M}^v W = aA^v. \tag{4}$$

This is a genuine one-site physical corner; it does not use the two-site gauge, a Gram identity, fusion data, or a renormalization map.

The matched positive copy in the two-site vertical decomposition similarly gives a matrix V and a scalar $b > 0$ such that

$$V^\dagger V = \mathbf{1}, \qquad V^\dagger \widetilde{M}^v_{[2]} V = bZA^vZ^{-1}. \tag{5}$$

The formal declarations for these two corners are

```
MPOTensor.BNTAlgebraTensorClause.TwoSiteExactSectorGauge.
exists_oneSite_raw_sector_corner,    exists_blockTwo_gauge_sector_corner.
```

3 The common unblocked reflected tail

Write $\mathfrak{m}_N(C; M)$ for a marked-chain coefficient with marked tensor C followed by a tail of N copies of M; the open bond indices and the two tail words are suppressed. Hermiticity of the matrix product density operators and (4) give

$$\mathfrak{m}_N(A; M) = \mathfrak{m}_N(A^{\mathrm{ref}\dagger}; M^\dagger), \tag{6}$$

with the bond indices interchanged and both tail words reversed on the right-hand side, as prescribed by Proposition 4.13.

For the two-site corner, one compresses the first two physical sites jointly through V and then leaves the next N sites unblocked. Hermiticity of the same length-$(N+2)$ density operator gives

$$\mathfrak{m}_N\big(\mathrm{gramDressing}(Z, A); M\big) = \mathfrak{m}_N(A^{\mathrm{ref}\dagger}; M^\dagger). \tag{7}$$

The right-hand side is exactly the target in (6); in particular, it contains the same unblocked $M^\dagger$ tail. The positive corner coefficients a and b cancel separately in the two reflection arguments. Hence, for every positive tail length and all open indices and tail words,

$$\mathfrak{m}_N\big(\mathrm{gramDressing}(Z, A); M\big) = \mathfrak{m}_N(A; M). \tag{8}$$

The mixed-prefix identity is formalized by

`MPOTensor.IsMPDO.`
`markedChainCoefficient_blockTwoPrefix_`
`gramDressing_eq_reflectedAdjoint.`

Its point is precisely that the two-site prefix is followed by the original one-site tensor, rather than by the blocked tensor.

4 Both marks lie in the one-site physical-letter span

Put

$$G = Z^\dagger Z. \tag{9}$$

The corner (4) realizes the raw marked tensor in the one-site physical-letter span by the two-sided pair (W, W). More explicitly, taking a matrix entry of $a^{-1}W^\dagger \widetilde{M}^v W = A^v$ expresses every horizontal slice of A as a linear combination of the physical letters of M.

The pair

$$(WG^\dagger,\, WG^{-1}) \tag{10}$$

realizes the Gram-dressed mark in the same one-site physical-letter span, because

$$\begin{aligned}(WG^\dagger)^\dagger \widetilde{M}^v (WG^{-1}) &= G\big(W^\dagger \widetilde{M}^v W\big)G^{-1} \\ &= aGA^vG^{-1} = a\,\mathrm{gramDressing}(Z, A)^v.\end{aligned} \tag{11}$$

Thus both sides of (8) satisfy the physical-letter premise of the literal CPSV Lemma L. This is the second essential feature of the repair: one does not need an identity-gauge corner inside the blocked physical space.

Lemma L now applies to (8). Literal CPSV canonical form supplies representative separation at a sufficiently large positive tail length, and therefore

$$\mathrm{gramDressing}(Z, A) = A. \tag{12}$$

Equivalently,

$$GA^vG^{-1} = A^v \qquad \text{for every vertical letter } v. \tag{13}$$

Since A is normal, its letters generate the full matrix algebra after a finite blocking. Equation (13) therefore places G in the scalar commutant. As $G = Z^\dagger Z$ is positive definite, there is a real number $\omega > 0$ such that

$$Z^\dagger Z = \omega \mathbf{1}. \tag{14}$$

Consequently

$$U = \omega^{-1/2} Z, \qquad U^\dagger U = \mathbf{1}, \qquad B^v = UA^vU^\dagger. \tag{15}$$

This proves the line 2057 unitary normalization without assuming a positive blocked raw corner, fusion coisometries, or renormalization maps.

5 Completion of Theorem 4.14

Applying the construction sector by sector gives simultaneous unitaries for the one-site and two-site vertical decompositions. The direct-sum conjugations, vertical retractions, and normalized restorations then give the two physical maps written in Appendix C.4, lines 2065–2085. The trace-restoring completion on discarded zero-sector complements is the local fix documented in `docs/paper-gaps/cpgsv17_vertical_isometry_zero_sector.tex`; its added terms vanish on the physical closures generated by M.

It follows that a tensor-attached BNT algebra clause, together with the standing literal CPSV canonical-form and MPDO hypotheses of the source, implies the trace-preserving completely positive renormalization identities of Definition 4.1. This is the unconditional implication Theorem 4.14(ii)$\Rightarrow$(i), formalized by

`MPOTensor.BNTAlgebraTensorClause.isRFPViaTS`.

The already formalized forward implication produces the corrected active-support fusion clause, and active-support fusion implies the tensor-attached algebra clause. Combining these directions gives

$$\begin{aligned} \mathrm{IsRFPViaTS}(M) &\iff \mathrm{HasBNTAlgebraTensorClause}(M), \\ \mathrm{IsRFPViaTS}(M) &\iff \mathrm{HasBNTFusionTensorClause}(M), \end{aligned} \tag{16}$$

under exactly the standing canonical-form and MPDO hypotheses. The second line uses the corrected active-support reading of statement (iii), including the previously documented empty-support and coisometry local fixes. No stronger horizontal canonical form and no extra reflected-target hypothesis is present in either equivalence. The corresponding formal declarations are

`MPOTensor.isRFPViaTS_iff_hasBNTAlgebraTensorClause`,
`MPOTensor.isRFPViaTS_iff_hasBNTFusionTensorClause`.

6 Why the former direct route stopped

The mixed-prefix proof does not invalidate the earlier obstruction analyses. It shows instead that they concerned an unnecessarily strong route: trying to construct a raw representative as a same-sided corner of the blocked physical tensor and then comparing two ordinary blocked chains.

6.1 The circular blocked raw corner

From the two-site gauge corner (5), the natural candidate for a blocked physical inclusion of A is scalar multiple of VZ. It is an isometry precisely when $Z^\dagger Z = \omega \mathbf{1}$. Indeed, after that identity is known, $\omega^{-1/2} VZ$ gives a positive same-sided raw corner. Conversely, a raw isometric corner of this form, compared with V, would prove the same Gram identity. Therefore this construction is circular if used to establish (14).

The repair avoids the circle. It uses the independent one-site corner W for the raw marked chain and uses V only for the two-site prefix. The two corners need not inhabit the same physical space because their reflection identities meet at the common unblocked tail.

6.2 The oblique blocked compression

The invertible gauge always gives the oblique maps

$$L = V(Z^{-1})^\dagger, \qquad R = VZ, \tag{17}$$

and hence

$$L^\dagger \widetilde{M}^v_{[2]} R = bA^v, \tag{18}$$

$$R^\dagger \widetilde{M}^v_{[2]} L = b(Z^\dagger Z)A^v(Z^\dagger Z)^{-1}. \tag{19}$$

This remains a useful physical-letter realization, but it is not a positive same-sided corner. Adjoint reflection exchanges L and R, so the opposite orientation is Gram-dressed. Thus the oblique blocked construction alone does not prove (12). The mixed-prefix proof uses neither this false inference nor a conversion of L and R into one map.

6.3 Fusion data cannot be imported into the converse

A local coisometry $F_{\alpha\beta}$ satisfying

$$F_{\alpha\beta}(A_\alpha A_\beta)^v F^\dagger_{\alpha\beta} = \bigoplus_\gamma \chi_{\alpha\beta\gamma} \otimes A^v_\gamma \tag{20}$$

would produce isometric blocked raw corners immediately. But such an $F_{\alpha\beta}$ is precisely the fusion datum in statement (iii). Using it to prove statement (ii)$\Rightarrow$(i) would be circular. The algebra clause supplies the closed-chain product law, not the local map in (20). The completed proof does not use fusion data in the converse direction.

7 Counterexamples delimiting the direct route

7.1 Closed chains and an invertible gauge are insufficient

Let the letters of A contain the identity and the matrix units of $\mathcal{M}_2(\mathbb{C})$, and put

$$X = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C^v = XA^vX^{-1}. \tag{21}$$

The tensors A and C have identical closed chains at every positive length. If that fact alone produced a square isometry Q and a positive scalar c with $Q^\dagger C^v Q = cA^v$, the identity letter would give $c = 1$. Since Q is then unitary,

$Q^\dagger X$ would commute with every matrix and hence would be scalar. This would force $X^\dagger X$ to be scalar, contrary to

$$X^\dagger X = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Likewise,

$$\mathrm{gramDressing}(X, A)^{v_0} = 4E_{01} \neq E_{01} = A^{v_0} \tag{22}$$

for a letter $A^{v_0} = E_{01}$.

These examples remain valid obstructions to deriving a blocked raw corner or a Gram identity from positive-length closed-chain equality and an invertible gauge alone. They do not contradict the completed tensor-attached theorem: they do not provide one MPO tensor satisfying the algebra clause, literal CPSV canonical form, and MPDO positivity from which both physical corners arise. The relevant formal declarations are

`MPSTensor.PositiveMinimalRealizationCounterexample.sameMPV₂Pos_does_not_force_positive_isometric_realization`, `gramDressing_gauge_ne_one`.

The same distinction explains the negative result for the per-block linear extension. A positive algebra automorphism of a full matrix algebra is a star automorphism and therefore has a unitary implementer by unitary Skolem–Noether. The generic nonunitary similarity makes the corresponding per-block extension nonpositive. Thus positivity of that extension cannot be inferred from closed chains alone. This fact is still useful, but the mixed-prefix proof does not require such positivity. The corresponding formal declarations are

`Matrix.exists_unitary_conj_of_positive_algEquiv_matrix`,
`MPSTensor.exists_unitary_conj_of_positive_perBlockLinearExtension`,
`MPSTensor.PositiveMinimalRealizationCounterexample.perBlockLinearExtension_not_positive`.

7.2 Terminal positivity and the bond-two test

For every retained one-site sector, tensor-attached MPDO positivity implies

$$T_\gamma = \sum_a A_\gamma^{(a,a)} \succeq 0. \tag{23}$$

This excludes the simplest nonunitary minimal-realization witness from the full theorem, because its terminal transfer is not Hermitian. The declaration

`MPOTensor.BNTAlgebraTensorClause.physTraceTransfer_verticalBNTMPO_posSemidef`

formalizes this necessary condition.

The bond-two singleton family provides a sharper diagnostic. Its undeformed member is an all-length MPDO with a genuine one-label tensor-attached algebra clause. A concrete nonunitary physical similarity preserves the algebra clause and literal horizontal CPSV canonical form, but its length-two MPO is not positive. Thus it fails the MPDO hypothesis. This example correctly shows that the positivity hypothesis is necessary; it is not a counterexample to Theorem 4.14. Its earlier role as evidence for a metric boundary is superseded by the mixed-prefix comparison. The formal tests are

```
MPOTensor.BondTwoSingletonBaseModel.
gaugeGram_eq_pos_smul_one_of_gaugeDeformedBaseMPO_isMPDO,
gaugeDeformedBaseMPO_gauge_not_isMPDO,
gaugeDeformedBaseMPOBNTAlgebraTensorClause,
gaugeDeformedBaseMPO_toMPSTensor_isCPSVCanonicalForm.
```

8 Verdict

The two-site sector comparison no longer stops at an invertible gauge. The one-site raw corner and the two-site gauge corner have reflection identities with the same unblocked tail. Their common reflected target equates the raw and Gram-dressed marked chains, and the pairs (W, W) and $(WG^\dagger, WG^{-1})$ place both marks in the one-site physical-letter span. Literal CPSV Lemma L yields gramDressing$(Z, A) = A$; normality then gives $Z^\dagger Z = \omega \mathbf{1}$ with $\omega > 0$, followed by unitary normalization and the physical renormalization maps.

This is a local repair of the source proof, not a change to its mathematical statement. It discharges the comparison formerly tracked in issues 5284, 4648, and 4645, establishes the unconditional line 2057 conclusion, and completes the tensor-attached algebra equivalence and the corrected active-support fusion equivalence associated with Theorem 4.14. The direct-route counterexamples above remain useful warnings: closed chains and invertible similarity do not by themselves produce a blocked raw corner. The completed proof succeeds because it compares a one-site corner with a mixed two-site prefix inside the same MPDO family.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.