Historical Note: Two-Layer Sector Refinement of CPSV16
`eq:II_ABasicTensors`

2026-05-13

Verdict: scope-restriction (historical).

1 Status

Historical proof archaeology, not an active gap. This note records the mismatch between a retired strict-modulus specialization and the repeated-copy BNT decomposition in CPSV16. The current `SectorDecomposition` and `IsBNTCanonicalForm` surfaces retain the raw copy weights, and `MPSTensor.fundamentalTheorem_proportional_canonicalForm` proves the corrected proportional theorem on active BNT families. No adapter from the retired specialization is a current task.

2 The source assertion

Cirac–Pérez-García–Schuch–Verstraete 2016 [CPGSV16] expresses any tensor A in canonical form in terms of its basis of normal tensors (BNT) A_j as

$$A^i = \bigoplus_{j=1}^{g} \bigoplus_{q=1}^{r_j} \mu_{j,q}\, X_{j,q}\, A_j^i\, X_{j,q}^{-1}, \qquad (\texttt{eq:II\_ABasicTensors})$$

where $\mu_{j,q} \in \mathbb{C}$ are *arbitrary* complex coefficients (no ordering or unit-modulus condition is imposed), $X_{j,q}$ are invertible gauge matrices, and A_j are the BNT normal blocks. The eventual linear independence of the BNT vectors $|V^{(N)}(A_j)\rangle$ is recorded separately in the review's definition of basis normal tensors [CPGSV21].

3 The retired specialization

An earlier formal route introduced a project-specific specialization of (`eq:II_ABasicTensors`). That route grouped equal-modulus blocks into sectors, extracted a common spectral factor, and imposed three hypotheses which are stronger than the source display:

1. **Strict spectral ordering.** The predicate requires the existence of spectral-level coefficients $\lambda_j \in \mathbb{C}$ (one per BNT sector j) with $|\lambda_0| > |\lambda_1| > \cdots > |\lambda_{g-1}|$ (formerly recorded as a strict-antitone condition on the chosen spectral levels). No such ordering is present in (eq:II_ABasicTensors).

2. **Unit-modulus within-sector factors.** Each weight $\mu_{j,q}$ is required to factor as $\mu_{j,q} = \lambda_j \cdot \nu_{j,q}$ with $|\nu_{j,q}| = 1$, recorded by an explicit unit-modulus factorization condition. The general form (eq:II_ABasicTensors) allows $|\mu_{j,q}|$ to vary arbitrarily within a sector. The unit-modulus factorization is conceptually related to the RFP characterization theorem of [CPGSV16] (lines 543–555), which is a strict sub-class of the general BNT.

3. **Dominant normalization.** The dominant spectral level satisfies $|\lambda_0| = 1$. The retired strict-modulus layer recorded this through the former field `mu_dom_norm_one`. This is historical terminology, not a live Lean declaration, and the condition is not imposed by (eq:II_ABasicTensors).

4 The surviving formal surface

The surviving surface separates the source display into two pieces. `SectorDecomposition` stores the basis blocks, their copy multiplicities, and the raw weights $\mu_{j,q}$ through the coefficient

$$P.\texttt{coeff}\, N\, j = \sum_q (P.\texttt{weight}\, j\, q)^N.$$

`IsBNTCanonicalForm` then records the BNT hypotheses used by the fundamental-theorem route: irreducibility, left canonical normalization, gauge-phase distinctness of equal-dimensional basis blocks, eventual linear independence of the basis MPVs, bounded weights, and existence of at least one unit-modulus weight.

This keeps the two-layer information visible without assuming strict ordering of spectral levels or a factorization $\mu_{j,q} = \lambda_j \nu_{j,q}$ as part of the BNT predicate. When such a factorization is needed, it should be stated as an additional theorem hypothesis rather than hidden in the canonical-form predicate.

5 Historical scope restriction and its resolution

The representative-family predicate `IsNormalCanonicalFormBNT` covers one chosen block per gauge-phase class and therefore does not itself store the repeated-copy data $r_j > 1$. The earlier route tried to bridge this restriction through a strict-modulus adapter. That adapter was retired.

The current `SectorDecomposition` surface directly permits $P.\texttt{copies}(j) > 1$, and the SectorBNT comparison recovers repeated weights from the power sums $\sum_q \mu_{j,q}^N$. Thus the old one-copy restriction is not an outstanding obligation of the proportional fundamental theorem. Its history is documented in `docs/paper-gaps/cpsv16_ft_one_copy_scope_restriction.tex`.

6 Formal declarations

The relevant surviving formal declarations are:

- `SectorDecomposition` - the raw two-layer sector surface (`TNLean/MPS/SharedInfra/SectorDecomposition`).
- `IsBNTCanonicalForm` - the BNT canonical-form predicate (`TNLean/MPS/FundamentalTheorem/SectorBNT/Basic`).
- `MPSTensor.IsBNTCanonicalForm.cross_overlap_basis_tendsto_zero`, `MPSTensor.IsBNTCanonicalForm.combined_family_eventually_li`, and `MPSTensor.IsBNTCanonicalForm.coeff_not_eventually_zero` in `TNLean/MPS/FundamentalTheorem/SectorBNT/Api`.
- `MPSTensor.fundamentalTheorem_proportional_canonicalForm` - the corrected proportional comparison for active BNT families (`TNLean/MPS/FundamentalTheorem/SectorBNT/FundamentalCoord`).

See also PR #1657 (branch `feat/mps-ft-path-beta-pr1-two-layer`) and the audit memo `docs/audits/2026-05-13_cpsv16_ft_path_beta_scout.md` §"PR 1.5 amendment".

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.