

The Arbitrary-Chain Topological Projector Recursion

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Verdict: open-gap (resolved).

1 The source statement

After Theorem 4.14, the source applies the fusion maps successively along an arbitrary chain. In the notation of lines 995–1010 of [CPGSV16], this gives

$$\rho^{(N)}(M) \simeq_{\text{loc. iso.}} \mu^{\otimes N} \tilde{U} Q_N \tilde{U}^\dagger, \quad (1)$$

where $\simeq_{\text{loc. iso.}}$ denotes the source’s local-isometry relation from Proposition 4.13, and $\tilde{U}$ is a sequential circuit assembled from the pairwise fusion maps.

Local fix (rectangular vertical map). The retained nonzero sectors may have smaller dimension than the original one-site space. Accordingly, the formal vertical map is a rectangular retained-row coisometry, and local equivalence means the pair of exact forward and reconstruction equations, not conjugation by a square unitary. This point is documented in `docs/paper-gaps/cpgsv17_vertical_isometry_zero_sector.tex`.

The same passage asserts

$$[\mu^{\otimes N}, \tilde{U} Q_N \tilde{U}^\dagger] = 0. \quad (2)$$

When the coefficients are independent of the positive chain length, the diagonal entries of every $\chi_{\alpha,\beta,\gamma}$ belong to $\{0, 1\}$. Spectral decomposition of the terminal matrices then yields the length-independent form stated at lines 1013–1016:

$$\rho^{(N)}(M) = \sum_i \lambda_i P_i^{(N)} e^{-H_N}, \quad [P_i^{(N)}, e^{-H_N}] = 0, \quad (3)$$

where H_N is a translationally invariant commuting nearest-neighbor Hamiltonian and the $P_i^{(N)}$ are orthogonal projections.

2 The formalized recursion and all-label density decomposition

For terminal orthogonal projections P_γ , one pairwise fusion step is the direct sum

$$Q_{\alpha,\beta} = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes P_\gamma. \quad (4)$$

Here each P_γ acts on the bond space of the vertically read tensor M_γ . The source terminal matrix $\text{tr}(M_\gamma)$ closes the horizontal operator leg and leaves precisely these two bond indices. In the formalization this matrix is the physical-trace transfer of M_γ . Length independence and positivity imply $\chi_{\alpha,\beta,\gamma} = I$ on each retained multiplicity space, hence $Q_{\alpha,\beta}$ is an orthogonal projection. This statement is formalized for `MPOTensor.BNTFusionCoisometryFamily` and for the tensor-attached clause `MPOTensor.BNTFusionTensorClause`.

The pairwise fusion identity therefore gives the well-typed equality

$$U_{\alpha,\beta} \text{tr}(M_\alpha M_\beta) U_{\alpha,\beta}^\dagger = \bigoplus_{\gamma} \chi_{\alpha,\beta,\gamma} \otimes \text{tr}(M_\gamma). \quad (5)$$

Moreover, $U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = I$ implies that $U_{\alpha,\beta}^\dagger Q_{\alpha,\beta} U_{\alpha,\beta}$ is an orthogonal projection whenever $Q_{\alpha,\beta}$ is an orthogonal projection. The older family `MPOTensor.BNTFusionIsometryFamily` proves the same one-layer statement under the additional full-support hypothesis. This is a specialization, not a construction with a different terminal space.

Local fix (terminal trace orientation). The active fusion map has the retained-row orientation $U_{\alpha,\beta} U_{\alpha,\beta}^\dagger = I$ recorded in `docs/paper-gaps/cpsv16_figure11_fusion_coisometry.tex`. Pull request #4640 temporarily interpreted the terminal trace as the one-site closed operator $O_1(M_\gamma)$. That operator closes the bond leg and acts on the common horizontal space. The terminal factor in the diagram defining Q_N has the opposite contraction: it closes the horizontal operator leg and acts on the bond space of M_γ . The two contractions are visible separately in `Papers/1606.00608/MPD0_0_L_A_.png` and `Papers/1606.00608/MPD0_Structure.png`. Equation (5) restores the latter contraction.

For a matrix product density operator, each terminal matrix in this orientation is now proved positive semidefinite:

$$\text{tr}(M_\gamma) \geq 0. \quad (6)$$

Indeed, at chain length one the retained vertical congruence preserves positivity. Restricting to one label γ and one diagonal coordinate q of its positive multiplicity matrix gives the principal block

$$(\mu_\gamma)_{q,q} \text{tr}(M_\gamma) \geq 0. \quad (7)$$

Since $(\mu_\gamma)_{q,q} > 0$, division by this scalar proves (6). This is formalized by `MPOTensor.BNTFusionTensorClause.physTraceTransfer_verticalBNTMPO_posSemidef`. It supplies the positivity needed for the spectral decomposition invoked at lines 1010–1016, but does not assert that the terminal matrices are projections.

The pairwise construction has now been iterated along an arbitrary finite chain with a fixed initial label. For total chain length $N \geq 1$, the appended-label list has length $N - 1$; the symbols Q_N and W_N below refer to this total chain length. A fusion history records the intermediate labels and multiplicity indices, its weight is the product of the selected diagonal entries of the structure matrices, and its terminal space is the bond space of its final label. The resulting recursively typed operator is

$$Q_N(P) = \bigoplus_h w(h) P_{\gamma(h)}. \quad (8)$$

The sequential product of active fusion coisometries is formalized together with its coisometry identity. It carries every letter of the left-associated product tensor to (8), with terminal block $P_\gamma = M_\gamma^{i,j}$. Summing over equal physical indices proves the fixed-label terminal-trace identity

$$W_N \operatorname{tr}(M_{\alpha,\beta}) W_N^\dagger = Q_N((\operatorname{tr}(M_\gamma))_\gamma). \quad (9)$$

Under length independence every history weight is one; consequently $Q_N(P)$ is an orthogonal projection whenever every terminal matrix P_γ is an orthogonal projection. The source embedding $\tilde{U}_N$ is $W_N^\dagger$, the adjoint of this retained-row coisometry; exact reverse fusion uses the active-support reconstruction of Appendix C.4, lines 2020–2029, as documented in `docs/paper-gaps/cpsv16_figure11_fusion_coisometry.tex`.

The vertical canonical form has now also been assembled over every initial label and every multiplicity coordinate. For a sitewise configuration $c = ((\alpha_n, q_n))_{n=0}^{N-1}$, put

$$m(c) = \prod_{n=0}^{N-1} (\mu_{\alpha_n})_{q_n, q_n}. \quad (10)$$

The first label is the initial label in the preceding recursion, and the remaining $N - 1$ labels are the appended list. If $W_{N,c}$ denotes the resulting retained-row sequential coisometry, set

$$Q_{N,c} = Q_N((\operatorname{tr}(M_\gamma))_\gamma). \quad (11)$$

The retained density block is

$$m(c) W_{N,c}^\dagger Q_{N,c} W_{N,c}. \quad (12)$$

Thus the embedding $\tilde{U}$ in (1) is the adjoint $W_{N,c}^\dagger$ of the coisometry used by the formalization. This orientation agrees with `docs/paper-gaps/cpsv16_figure11_fusion_coisometry.tex`.

Let V be the rectangular vertical coisometry of Proposition 4.13 and let R_N be the direct sum of (12) over all configurations c . For every $N > 0$, the formalization proves both

$$V^{\otimes N} \rho^{(N)}(M) (V^{\otimes N})^\dagger = R_N \quad \text{and} \quad ((V^\dagger)^{\otimes N}) R_N V^{\otimes N} = \rho^{(N)}(M). \quad (13)$$

Local fix (rectangular vertical map). The reverse identity uses the exact one-site reconstruction from the retained nonzero sectors in Proposition 4.13 and Appendix C.4, lines 2020–2029, not a full-support or square-unitary assumption. The rectangular zero-sector point is documented in `docs/paper-gaps/cpgs v17_vertical_isometry_zero_sector.tex`. Equation (10) is exactly the selected diagonal coefficient of $\mu^{\otimes N}$. Hence (13) proves the density identity at line 999 with the source chain length N and no additional compatibility hypothesis.

The assembled operator is `MPOTensor.BNTFusionTensorClause.topologicalDensityOperatorSucc`. Its argument is one less than the positive source chain length. The source-facing existence theorem is `MPOTensor.exists_topologicalDensityDecomposition_of_isRFPViaTS`; it chooses one vertical decomposition and proves both equalities in (13) for every positive N .

3 The all-label commutator

On the same all-label coordinates, let

$$A_N = \bigoplus_c m(c) I_c, \quad T_N = \bigoplus_c W_{N,c}^\dagger Q_{N,c} W_{N,c}. \quad (14)$$

The operator in (13) is $R_N = A_N T_N$. Within every copy-configuration block, A_N is a scalar multiple of the identity. Hence direct-sum multiplication gives

$$A_N T_N = T_N A_N. \quad (15)$$

This proves (2) for every positive chain length, without length independence or a caller-supplied compatibility condition. The two factors are `MPOTensor.BNTFusionTensorClause.topologicalMultiplicityWeightFactorSucc` and `MPOTensor.BNTFusionTensorClause.topologicalRecursiveFactorSucc`. The source-facing theorem `MPOTensor.exists_topologicalDensityDecomposition_and_factorCommutator_of_isRFPViaTS` chooses the same fusion clause for (13) and (15).

4 The terminal spectral refinement

Fix the BNT fusion clause selected from the horizontal-canonical MPDO and its renormalization fixed-point data. The length-independence hypothesis is stated on the positive trace-power coefficients of this fixed clause; it is not an existential choice of a different fusion clause.

For every terminal label γ , let

$$K_\gamma = \text{tr}(M_\gamma) = \sum_k \lambda_{\gamma,k} E_{\gamma,k}.$$

One-site positivity of the MPDO, restricted to a fixed BNT copy and divided by its strictly positive multiplicity weight, proves $K_\gamma \geq 0$. Thus $\lambda_{\gamma,k} \geq 0$, and the $E_{\gamma,k}$ are rank-one orthogonal projections. Extend $E_{\gamma,k}$ by zero on the other terminal labels and transport it through the recursive fusion histories and the adjoint sequential coisometries. After the all-label assembly this gives $P_{\gamma,k}^{(N)}$ with

$$(P_{\gamma,k}^{(N)})^\dagger = P_{\gamma,k}^{(N)}, \quad (P_{\gamma,k}^{(N)})^2 = P_{\gamma,k}^{(N)},$$

and

$$P_{\gamma,k}^{(N)} P_{\delta,\ell}^{(N)} = 0 \quad \text{when } (\gamma, k) \neq (\delta, \ell).$$

For every positive chain length,

$$T_N = \sum_{\gamma,k} \lambda_{\gamma,k} P_{\gamma,k}^{(N)}, \quad [A_N, P_{\gamma,k}^{(N)}] = 0, \quad (16)$$

and hence

$$R_N = \sum_{\gamma,k} \lambda_{\gamma,k} A_N P_{\gamma,k}^{(N)}. \quad (17)$$

This is formalized by `MPOTensor.terminalSpectralProjectorRefinement_of_isRFPViatS`.

5 Remaining physical-space construction

The all-label density identity (1), its commutator (2), and the terminal spectral refinement (16) are now combined with an explicit commuting nearest-neighbor Hamiltonian on the retained vertical space. For every chain length $N \geq 2$, the two-site interaction is the logarithmic multiplicity energy on its first site and the identity on its second site. Its periodic translates are diagonal and commute, and their negative exponential is $\mu^{\otimes N}$. The terminal spectral family has cardinality at most the physical one-site dimension d and is extended by zero to exactly d indices. This proves the retained-coordinate analogue of (3).

The adjoint retained-row map reconstructs the physical density operator from that analogue. It does not yet give physical projectors and a finite physical-space Hamiltonian because the rectangular vertical map can have a nontrivial complementary zero sector. The necessary complementary-subspace extension is recorded in `docs/paper-gaps/cpsv16_topological_gibbs_physical_complement.tex`.

Independently, the cited source defines the local term on two spins but gives no separate convention for a one-site periodic chain. This boundary is recorded in `docs/paper-gaps/cpsv16_topological_gibbs_length_one.tex`.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.