The Physical Complement in the Topological Gibbs Decomposition

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Verdict: open-gap (resolved).

1 Status

Resolved for the source-defined domain of chain lengths $L \geq 2$. The generic fusion-clause theorem is

`MPOTensor.BNTFusionTensorClause.hasPhysicalTopologicalGibbsDecomposition`,

and the literal CPSV fixed-point capstone is

`MPOTensor.physicalTopologicalGibbsDecomposition_of_isRFPViaTS_of_cpsvCanonicalForm`.

The earlier normalized BNT-refined corollary

`MPOTensor.physicalTopologicalGibbsDecomposition_of_isRFPViaTS`

remains available. These theorems construct the finite physical-complement extension, transport the retained projectors, and prove the physical density formula. The separate undefined length-one convention remains documented in `docs/paper-gaps/cpsv16_topological_gibbs_length_one.tex`.

2 Source statement

The theorem following Theorem 4.14 in [CPGSV16, lines 1013–1016] states an equality on the physical chain:

$$\rho^{(N)}(M) = \sum_{i=1}^{d} \lambda_i P_i^{(N)} e^{-H_N}, \qquad [P_i^{(N)}, e^{-H_N}] = 0. \tag{1}$$

Here the $P_i^{(N)}$ are orthogonal projections on the physical Hilbert space, and $H_N = \sum_j h_{j,j+1}$ is a translationally invariant commuting nearest-neighbor Hamiltonian on that space.

The preceding derivation first applies the vertical canonical form from Proposition 4.13 and then suppresses the resulting local change of coordinates. The formal vertical map is a retained-row coisometry

$$V : \mathbb{C}^d \longrightarrow \mathbb{C}^s, \qquad VV^\dagger = I_s, \qquad V^\dagger V = E.$$

The projection E need not be the identity because the original physical space may contain a zero sector. This is documented in `docs/paper-gaps/cpgsv17_vertical_isometry_zero_sector.tex`.

3 Retained-coordinate theorem

Let k be the diagonal logarithmic multiplicity energy on $\mathbb{C}^s$. For every $N \geq 2$, the formalized two-site interaction on the retained space is

$$h = k \otimes I.$$

Its periodic translates commute and satisfy

$$\exp\left(-\sum_j h_{j,j+1}\right) = \mu^{\otimes N}.$$

The terminal spectral projections on the retained chain commute with this factor and give

$$R_N = \sum_i \lambda_i P_{i,\mathrm{ret}}^{(N)} e^{-H_{N,\mathrm{ret}}}. \tag{2}$$

Applying $(V^\dagger)^{\otimes N}$ and $V^{\otimes N}$ to the two sides of (2) reconstructs $\rho^{(N)}(M)$ exactly.

This reconstruction is not equation (1). The Hamiltonian $H_{N,\mathrm{ret}}$ and the projections $P_{i,\mathrm{ret}}^{(N)}$ act on $(\mathbb{C}^s)^{\otimes N}$, whereas the operators in equation (1) must act on $(\mathbb{C}^d)^{\otimes N}$.

4 Failure of direct compression and a finite physical extension

Direct transport of the retained Gibbs factor gives

$$V^\dagger e^{-k} V.$$

This matrix vanishes on $(I-E)\mathbb{C}^d$ and is singular whenever $E \neq I$. It therefore cannot equal the exponential of a finite matrix on the physical one-site space.

A viable physical extension begins with

$$k_{\mathrm{phys}} = V^\dagger k V, \qquad e^{-k_{\mathrm{phys}}} = V^\dagger e^{-k} V + (I - E). \tag{3}$$

The complementary summand supplies finite Gibbs weight one. If $H_{L,\mathrm{phys}}$ is the sum of the periodic translates of $k_{\mathrm{phys}} \otimes I$, then

$$e^{-H_{L,\mathrm{phys}}} = \bigotimes_{j=1}^{L} \left(V^\dagger e^{-k} V + I - E\right). \tag{4}$$

This differs from the false global equality

$$e^{-H_{L,\mathrm{phys}}} = (V^\dagger)^{\otimes L} e^{-H_{L,\mathrm{ret}}} V^{\otimes L},$$

whose right-hand side vanishes on every complementary or mixed sector.

The physical projectors are naturally defined by

$$\widehat{P}_i^{(N)} = (V^\dagger)^{\otimes N} P_{i,\mathrm{ret}}^{(N)} V^{\otimes N}.$$

They are supported on $E^{\otimes N}$ and therefore annihilate every complementary and mixed-sector contribution. The coisometry identity gives their projection identities, while equation (4) reduces their commutation and the physical density formula to the retained equalities. The resolution below formalizes this finite physical extension.

5 Resolution

The formalization resolves the gap by the following steps.

1. It proves equation (3) from $VV^\dagger = I_s$.
2. It defines a fixed physical two-site interaction from k_{phys} and proves the periodic exponential identity for every chain length at least two.
3. It transports the retained terminal projections to the physical chain and proves their projection, orthogonality, and commutation identities.
4. It combines these identities with exact vertical reconstruction to obtain equation (1).

The separate one-site ambiguity in the two-site periodic interaction is recorded in `docs/paper-gaps/cpsv16_topological_gibbs_length_one.tex`; it is not resolved by the physical-complement construction.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.