The Domain Boundary of the Topological Gibbs Decomposition

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Verdict: local-correction (resolved).

1 Source range

Theorem 4.15 in [CPGSV16, lines 999–1016] writes

$$\rho^{(N)}(M) = \sum_{i=1}^{d} \lambda_i P_i^{(N)} e^{-H_N}, \qquad [P_i^{(N)}, e^{-H_N}] = 0,$$

where $H_N = \sum_i h_{i,i+1}$ is a translationally invariant commuting nearest-neighbor Hamiltonian. Definition 4.8 [CPGSV16, lines 829–847] defines the local interaction by letting a matrix h act on the first two spins and translating that two-spin operator around the chain. This construction has domain $N \geq 2$: the periodic identification $N+1=1$ identifies site labels, but does not identify the two tensor factors on which h acts.

2 The retained-coordinate range

Let μ be the strictly positive diagonal one-site multiplicity matrix in the retained coordinates. Define the two-site matrix

$$h = \operatorname{diag}(-\log \mu) \otimes I.$$

For every chain length $N \geq 2$, its periodic translates are well-defined, diagonal, Hermitian, and pairwise commuting. Every site occurs once as the first site of a translated term, hence

$$\exp\left(-\sum_i h_{i,i+1}\right) = \mu^{\otimes N}.$$

Combining this equality with the terminal spectral refinement proves the retained-coordinate Gibbs decomposition for every $N \geq 2$. Applying the adjoint retained-row map reconstructs the physical density operator, but does not yet supply a physical-space Hamiltonian and physical-space projectors. That independent gap is recorded in `docs/paper-gaps/cpsv16_topological_gibbs_physical_complement.tex`.

The terminal spectral sectors are naturally indexed by pairs (γ, k). Positivity of every multiplicity shows that one such sector embeds into each retained copy, and the coisometry rank bound gives

$$\#\{(\gamma, k)\} \leq d.$$

Choose the resulting injection into $\{1, \ldots, d\}$ and extend both the terminal eigenvalues and projectors by zero. The added zero matrices are self-adjoint idempotents and commute with the Gibbs factor, so this gives exactly d summands without asserting that the number of nonzero terminal sectors equals d.

3 Printed domain boundary

At $N = 1$, Definition 4.8 supplies neither an operator on the one-site Hilbert space nor a convention for the coincident periodic bond $h_{1,1}$. In particular, it gives no map from two-site operators to one-site operators and no multiplicity for the self-edge. Replacing h by $\operatorname{diag}(-\log \mu)$, tracing out a tensor factor, restricting to a diagonal subspace, or otherwise collapsing the two tensor factors would add a convention absent from the cited passage.

The source construction therefore determines the theorem on the domain

$$N \geq 2.$$

Theorem 4.15 is printed without stating this lower bound, but its $N = 1$ instance is undefined unless an author or erratum specifies both H_1 and the coincident-edge convention. The formal retained- and physical-coordinate theorems therefore assert no length-one case. The physical-coordinate complement is now complete on $N \geq 2$; see `docs/paper-gaps/cpsv16_topological_gibbs_physical_complement.tex`. That completion is independent of, and does not supply, the missing length-one convention.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.