Closure Record for the Strong-Subadditivity Equality Characterization and Hayashi–Markov Decomposition

2026-07-28

Verdict: open-gap (resolved).

1 The assertion and its current formal status

Equality in strong subadditivity,

$$S(\rho_{ABC}) + S(\rho_B) = S(\rho_{AB}) + S(\rho_{BC}), \tag{SSA$_=$}$$

is characterized by a structural decomposition of the middle subsystem. The characterization states that (SSA$_=$) holds if and only if, after a unitary change of basis on $\mathcal{H}_B$, the middle space decomposes as a finite orthogonal direct sum

$$\mathcal{H}_B \;\cong\; \bigoplus_j \mathcal{H}_{B_j^L} \otimes \mathcal{H}_{B_j^R}, \tag{1}$$

and in this basis the state is block diagonal,

$$\rho_{ABC} \;\cong\; \bigoplus_j p_j \, \rho_{AB_j^L} \otimes \rho_{B_j^R C}, \tag{QMC}$$

with probabilities $p_j \geq 0$, $\sum_j p_j = 1$, and density operators $\rho_{AB_j^L}$, $\rho_{B_j^R C}$ on the indicated factors. This is the quantum-Markov-chain structure of the tripartite state [1, 2, 3].

The biconditional (SSA$_=$) $\Leftrightarrow$ (QMC) is now proved in both directions.[1] There are no axiom declarations in this characterization.

[1] The forward implication is `Matrix.hayashi_ssa_equality_characterization_forward` in `TNLean/Analysis/EntropyMarkovForward.lean`, with the established

The forward proof first obtains the ambient HJPW identity

$$W^\dagger \rho_{ABC} W = 0_\perp \oplus \bigoplus_j \omega_j \otimes \widehat{\rho}_j, \qquad W = \mathbf{1}_A \otimes (U_B \otimes \mathbf{1}_C). \tag{2}$$

The supported matrices $\widehat{\rho}_j$ are the sector output states: they are positive, have trace one, and have the prescribed common-factor marginals. The complementary blocks are zero. After reversing each middle fibre from the recovery order $B_j^R \otimes B_j^L$ to the HJPW order $B_j^L \otimes B_j^R$, the Hayashi unitary is chosen with the adjoint orientation $V_B = U_B^\dagger$, so $V \rho_{ABC} V^\dagger = W^\dagger \rho_{ABC} W$.

For every ambient sector, including the one-dimensional complementary sectors, set

$$p_j = \operatorname{Re}\operatorname{tr}\omega_j. \tag{3}$$

Positivity gives $p_j \geq 0$, and the ambient trace identity gives $\sum_j p_j = 1$. On every supported sector, the right density operator is the sector output state $\widehat{\rho}_j$. When $p_j > 0$, the left density operator is ω_j / p_j. Total positive-semidefinite normalization is applied only when the final `HayashiMarkovDecomposition` is formed. If a supported sector has $p_j = 0$, this normalization chooses a left filler while preserving $\widehat{\rho}_j$ on the right. On a complementary sector it may choose fillers on both sides. In either case the factor p_j makes every filler invisible; fillers do not occur in the raw identity (2).

2 The two proved implications

Forward direction. Equality in strong subadditivity gives the product-reference recovery identity on the supported input, the invariant conditional family, its common invariant block form, the ambient bipartite decomposition, and the pure-ancilla block action. Equations (11), (14), and (15) then give (2). The probability and normalization argument above converts that identity into the quantum-Markov decomposition (QMC). This implication is proved by `Matrix.hayashi_ssa_equality_characterization_forward`.

root name `hayashi_ssa_equality_characterization_forward` in `TNLean/Entropy/SSAEqualityCharacterization`. The reverse implication is `hayashi_ssa_equality_characterization_reverse` in `TNLean/Analysis/EntropyMarkovReverse.lean`; `hayashi_ssa_equality_characterization` combines the two. The equality predicate is `IsSSAEquality` in `TNLean/Analysis/Entropy`, and the right-hand structure is `HayashiMarkovDecomposition`.

Reverse direction. For a state of the form (QMC),

$$S\left(\bigoplus_j p_j\omega_j\right) = H(\{p_j\}) + \sum_j p_j S(\omega_j), \tag{4}$$

$$S(\omega \otimes \tau) = S(\omega) + S(\tau). \tag{5}$$

Applying these identities to the state and its three marginals makes the four terms in ($\mathrm{SSA}_=$) cancel. This implication is proved by `hayashi_ssa_equality_characterization_reverse`.

The public biconditional `hayashi_ssa_equality_characterization` depends only on these two proved directions.

3 Trace-preserving Petz channel now available

The support formula for the Petz transpose map associated with a right partial trace is now formalized for every positive semidefinite reference σ. Writing $\tau = \mathrm{tr}_R\,\sigma$, the map is

$$\mathcal{R}_\sigma(X) = \sqrt{\sigma}\,(\tau_{\mathrm{supp}}^{-1/2} X \tau_{\mathrm{supp}}^{-1/2} \otimes \mathbf{1}_R)\sqrt{\sigma}. \tag{6}$$

It is proved completely positive in rectangular Kraus form, satisfies $\mathcal{R}_\sigma(\tau) = \sigma$, and obeys the support trace identity $\mathrm{tr}(\mathcal{R}_\sigma(X)) = \mathrm{tr}(P_\tau X)$. [2] This is exactly the formula stated on the support in [2, Theorem 3, equation (8)].

For a normalized reference, let $Q_\tau = \mathbf{1} - P_\tau$ and $\omega_R = \mathrm{tr}_L\,\sigma$. The complementary term

$$\mathcal{C}_\sigma(X) = Q_\tau X Q_\tau \otimes \omega_R \tag{7}$$

is completely positive and contributes trace $\mathrm{tr}(Q_\tau X)$. Hence $\widehat{\mathcal{R}}_\sigma = \mathcal{R}_\sigma + \mathcal{C}_\sigma$ is a trace-preserving completely positive map. It agrees with the support formula when $P_\tau X P_\tau = X$ and still recovers the reference matrix from its marginal. [3]

The source support condition is also now connected to this completed channel. If positive semidefinite matrices ρ and σ satisfy $\ker\sigma \subseteq \ker\rho$, then

[2]Formal declarations: `Matrix.partialTraceRightPetzMap`, `Matrix.partialTraceRightPetzMap_isKrausCP`, `Matrix.partialTraceRightPetzMap_trace`, and `Matrix.partialTraceRightPetzMap_partialTraceRight` in `TNLean/Channel/PetzRecovery.lean`.

[3]Formal declarations: `Matrix.partialTraceRightPetzComplementMap`, `Matrix.partialTraceRightPetzChannel`, `Matrix.partialTraceRightPetzChannel_isKrausCPTP`, `Matrix.partialTraceRightPetzChannel_apply_of_supported`, and `Matrix.partialTraceRightPetzChannel_partialTraceRight` in `TNLean/Channel/PetzRecovery.lean`.

the same inclusion holds after the right partial trace. If P_τ is the support projection of $\tau = \mathrm{tr}_R\,\sigma$, this gives

$$P_\tau(\mathrm{tr}_R\,\rho)P_\tau = \mathrm{tr}_R\,\rho. \tag{8}$$

Consequently the complementary term vanishes on $\mathrm{tr}_R\,\rho$, and

$$\widehat{\mathcal{R}}_\sigma(\mathrm{tr}_R\,\rho) = \mathcal{R}_\sigma(\mathrm{tr}_R\,\rho). \tag{9}$$

[4] Thus the choice of trace-preserving extension away from the support is no longer part of the forward equality gap.

The tensor-product identity required to identify the pre-twirl pair is also available without invertibility. If ρ and σ are positive semidefinite, $\ker\sigma \subseteq \ker\rho$, and τ is positive semidefinite with $\mathrm{tr}\,\tau = 1$, then

$$D(\rho\otimes\tau\|\sigma\otimes\tau) = D(\rho\|\sigma). \tag{10}$$

The proof expands the logarithm of a singular tensor product through the three support projections; support absorption and $\mathrm{tr}\,\tau = 1$ remove the projection and ancilla terms. [5] This identity is a prerequisite for transporting equality through the Weyl-twirl proof of data processing. It does not prove that equality yields recovery.

The scalar equality can now be transported from the partial-trace pair to every individual Weyl summand. Fix a primitive d_R-th root of unity, put $U_{ab} = \mathbf{1}_L \otimes W(a,b)$, and write

$$\overline{X} = \frac{1}{d_R^2}\sum_{c,e} U_{ce} X U_{ce}^\dagger = (\mathrm{tr}_R\,X)\otimes(d_R^{-1}\mathbf{1}_R). \tag{11}$$

If ρ and σ are positive semidefinite, $\ker\sigma \subseteq \ker\rho$, and partial-trace data processing is saturated, then for every a, b,

$$D(\overline{\rho}\|\overline{\sigma}) = D(U_{ab}\rho U_{ab}^\dagger\|U_{ab}\sigma U_{ab}^\dagger). \tag{12}$$

Indeed, the twirl identity, support-domain ancilla additivity, and the inherited support inclusion for the marginals identify the left-hand side with $D(\mathrm{tr}_R\,\rho\|\,\mathrm{tr}_R\,\sigma)$; unitary invariance identifies the right-hand side with

[4] Formal declarations: `Matrix.IsHermitian.supportProj_mul_mul_supportProj_of_mulVec_kernel_le` and `Matrix.partialTraceRightPetzChannel_apply_partialTraceRight_of_support` in `TNLean/Channel/Schwarz/PetzRecoverySupport.lean`.

[5] Formal declaration: `quantumRelativeEntropy_kronecker_support` in `TNLean/Channel/Schwarz/PetzEqualityPrerequisites.lean`.

$D(\rho\|\sigma)$. [6] This is only equality of the scalar relative entropies. It neither characterizes equality in the joint-convexity inequality nor supplies a Petz intertwining or recovery identity.

Summing the preceding equality with the uniform Weyl weights gives the scalar equality associated with the finite Jensen step:

$$D(\overline{\rho}\|\overline{\sigma}) = \frac{1}{d_R^2} \sum_{c,e} D(U_{ce}\rho U_{ce}^\dagger \| U_{ce}\sigma U_{ce}^\dagger). \tag{13}$$

Indeed, unitary invariance makes every summand equal to $D(\rho\|\sigma)$, while the preceding Weyl-summand equality gives the same value on the left. The d_R^2 weights sum to one. [7] This statement does not characterize equality in joint convexity and does not imply an operator intertwining or Petz recovery identity.

For positive definite inputs, the fixed-resolvent algebra of the equality case is now available for the Weyl family. With $A_g = d_R^{-2} U_g \rho U_g^\dagger$, $B_g = d_R^{-2} U_g \sigma U_g^\dagger$, and

$$T_g(X) = A_g X + tXB_g, \qquad T(X) = \Big(\sum_g A_g\Big)X + tX\Big(\sum_g B_g\Big),$$

put $\Lambda_t = T^{-1}(\sum_g B_g)$. For every $t > 0$, direct expansion of the residuals $R_g = B_g - T_g(\Lambda_t)$ gives the exact identity

$$\begin{aligned}\sum_g \Big\| T_g^{-1/2}(B_g) - T_g^{1/2}(\Lambda_t) \Big\|_{\mathrm{HS}}^2 = {} & \sum_g \mathrm{Re}\langle B_g, T_g^{-1}(B_g)\rangle_{\mathrm{HS}} \\ & - \mathrm{Re}\left\langle \sum_g B_g, T^{-1}\Big(\sum_g B_g\Big)\right\rangle_{\mathrm{HS}}. \end{aligned} \tag{14}$$

Consequently, if the right-hand side vanishes, then $T_g^{-1}(B_g) = \Lambda_t$ for every g, in particular for the identity Weyl element. The squared-defect identity is the Appendix calculation of Jenčová–Ruskai, arXiv:0903.2895v4, lines 1313–1343; the resulting common-resolvent equality is in §4, lines 652–674. [8] The

[6] Formal declaration: `quantumRelativeEntropy_weyl_average_eq_summand_of_partialTraceRight_eq` in `TNLean/Channel/Schwarz/PetzEqualityPrerequisites.lean`.

[7] Formal declaration: `quantumRelativeEntropy_weyl_jensen_eq_of_partialTraceRight_eq` in `TNLean/Channel/Schwarz/PetzEqualityPrerequisites.lean`.

[8] Formal declarations: `Matrix.weyl_resolvent_defect_identity` and `Matrix.weyl_resolvent_eq_of_defect_eq_zero` in `TNLean/Channel/Schwarz/PetzEqualityPrerequisites.lean`.

positive-definite analytic connection is now available. The scalar spectral identity is integrated exactly, and the finite Weyl gap is written as

$$\sum_g D(A_g\|B_g) - D(A\|B) = \int_0^\infty \frac{\mathrm{Def}_A(t) + t\,\mathrm{Def}_B(t)}{1+t}\,dt.$$

There is an essential normalization point. Saturation of the finite Weyl Jensen inequality gives d_R^{-2} times each relative entropy, whereas the displayed gap uses the weighted matrices $A_g = d_R^{-2} U_g \rho U_g^\dagger$ and $B_g = d_R^{-2} U_g \sigma U_g^\dagger$. Positive homogeneity, $D(cA\|cB) = cD(A\|B)$ for $c > 0$, identifies these two expressions. Consequently equality under the partial trace makes this precise weighted gap vanish; no separate gap hypothesis is needed. Both defects are nonnegative and continuous on $(0, \infty)$, while the full integrand is integrable there. Hence a zero gap first gives almost-everywhere vanishing and then, by continuity, pointwise vanishing for every $t > 0$. Since $t/(1 + t) > 0$ on this domain, the source-B defect vanishes, and the fixed-resolvent theorem gives $T_g^{-1}(B_g) = T^{-1}(B)$ for every Weyl index. [9] The positive-definite inverse-square-root passage is now also complete. The source-B resolvent factors through the relative modular matrix $\Delta_{A,B} = A \otimes (B^{-1})^\mathsf{T}$. Equality of its shifted resolvents on the vectorized identity passes through the positive square-root functional calculus, giving

$$\sqrt{\rho}\,\sigma^{-1/2} = \sqrt{\overline{\rho}}\,\overline{\sigma}^{-1/2}. \tag{15}$$

Taking adjoint products and cancelling $\sqrt{\sigma}$ gives

$$\sqrt{\sigma}\,\overline{\sigma}^{-1/2}\,\overline{\rho}\,\overline{\sigma}^{-1/2}\,\sqrt{\sigma} = \rho. \tag{16}$$

This is the positive-square-root specialization of the relative-modular analytic-continuation argument in Jenčová–Ruskai, arXiv:0903.2895v4, lines 658–680. The formal proof uses the equivalent Löwner integral representation of the power $p = 1/2$. [10] Thus equality under the partial trace

[9] Formal declarations: `Matrix.quantumRelativeEntropy_smul_posDef`, `Matrix.weylRelativeEntropyGap_eq_zero_of_partialTraceRight_eq`, `Matrix.quantumRelativeEntropy_resolvent_integral`, `Matrix.weyl_relativeEntropy_gap_integral`, `Matrix.weyl_sourceB_defect_eq_zero_of_gap_eq_zero`, `Matrix.weyl_resolvent_eq_of_relativeEntropy_gap_eq_zero`, and `Matrix.weyl_resolvent_eq_of_partialTraceRight_eq` in `TNLean/Analysis/RelativeEntropyResolventIntegral.lean` and `TNLean/Channel/Schwarz/WeylRelativeEntropyIntegral.lean`.

[10] Formal declarations: `Matrix.sqrt_mulVec_eq_of_resolvent_mulVec_eq`, `Matrix.mulVec_sqrt_mulVec_eq_of_mulVec_resolvent_mulVec_eq`, `Matrix.sourceB_resolvent_eq_relativeModular`, and `Matrix.relativeModular_sqrt_mulVec_vec_one` in

implies raw Petz recovery for positive definite inputs. The support-domain argument below extends this passage to singular supports.

Jenčová–Ruskai extend the definition and convexity argument directly to positive semidefinite pairs with $\ker B \subseteq \ker A$; see arXiv:0903.2895v4, lines 717–720. Their singular equality argument then obtains the common relative-modular resolvent on the reference support and passes through functional calculus; see lines 766–793. The support-domain spectral integral is now also proved. For positive semidefinite A, B with $\ker B \subseteq \ker A$, its finite spectral integrand is integrable on $(0, \infty)$ and has integral $D(A\|B)$. This proves the spectral content of (J1), (intspec), and (intAB) together with the support convention at lines 717–720. [11] The finite spectral integrand is now identified pointwise with the support-projected left–right quadratic form: the source in its first quadratic term is AP_B. Under the inclusion $\ker B \subseteq \ker A$, one has $AP_B = A$, so this term is the corresponding quadratic form with the unprojected source A. For a single pair, the source-B support-generalized-inverse solution is now identified with the projected shifted relative-modular vector

$$(\mathbf{1} \otimes P_B^{\mathsf{T}})\big(t\mathbf{1} + A \otimes (B^{+})^{\mathsf{T}}\big)^{-1} \operatorname{vec}(\mathbf{1}^{\mathsf{T}}).$$

The corresponding source-B quadratic forms are therefore equal. [12] For a finite family (A_i, B_i), both fixed-parameter quadratic defects are now nonnegative. Under $\ker B_i \subseteq \ker A_i$, the source-$A$ defect is

$$\sum_i Q^{A}_{A_i,B_i}(t) - Q^{A}_{\sum_i A_i, \sum_i B_i}(t) \geq 0,$$

where the source is the unprojected matrix A_i; the corresponding source-B inequality requires no kernel inclusion:

$$\sum_i Q^{B}_{A_i,B_i}(t) - Q^{B}_{\sum_i A_i, \sum_i B_i}(t) \geq 0.$$

`TNLean/Analysis/ResolventFunctionalCalculus.lean`; and `Matrix.weyl_sqrt_ratio_eq_of_partialTraceRight_eq`, `Matrix.weyl_identity_sandwich_of_partialTraceRight_eq`, and `Matrix.partialTraceRightPetzMap_eq_of_relativeEntropy_eq_posDef` in `TNLean/Channel/Schwarz/WeylRelativeEntropyIntegral.lean`.

[11] Formal declarations: `Matrix.quantumRelativeEntropy_spectral` and `Matrix.supportRelativeEntropySpectralIntegrand_integrableOn_and_integral_eq_quantumRelativeEntropy` in `TNLean/Analysis/RelativeEntropySupportIntegral.lean`, and `Matrix.supportRelativeEntropyLeftRightIntegrand_eq_spectral` and `Matrix.rawSupportSourceAQuadratic_sub_trace_add_sourceB_eq_spectral` in `TNLean/Analysis/RelativeEntropySupportLeftRightQuadratic.lean`.

[12] Formal declarations: `Matrix.supportLeftRightSupportInv_mulVec_sourceB_eq_projected_relativeModular` and `Matrix.supportSourceBQuadratic_eq_projected_relativeModular` in `TNLean/Channel/Schwarz/SupportLeftRightRelativeModular.lean`.

The kernel inclusions for the summands imply the same inclusion for the summed pair. [13] These are fixed-t algebraic inequalities. The remaining analytic passage now identifies the finite-family relative-entropy gap with the integral of the two defects:

$$\sum_i D(A_i\|B_i) - D\left(\sum_i A_i \middle\| \sum_i B_i\right) = \int_0^\infty \frac{\mathrm{Def}_A(t) + t\,\mathrm{Def}_B(t)}{1+t}\,dt.$$

The gap integrand is continuous on the positive half-line and integrable there. [14] The two defect inequalities make this integrand nonnegative. If the relative entropy of the summed pair equals the sum of the individual relative entropies, its integral is zero. Almost-everywhere vanishing and continuity then give pointwise vanishing for every $t > 0$; in particular, the source-B defect vanishes there. [15] The residual identity now converts this real-valued defect equality into the common support-generalized-inverse solution. The local right-support projection absorbs both the support projection of the local left–right operator and the right-support projection of the summed reference matrix. Consequently the local and summed shifted relative-modular resolvents agree after the local right-support projection, for every summand and every positive parameter. [16] This generic result is now specialized to the finite Weyl family produced by partial-trace saturation. For the unweighted conjugates $A_g = U_g\rho U_g^\dagger$ and $B_g = U_g\sigma U_g^\dagger$, positive-scalar homogeneity on the support domain converts the weighted Jensen equality into

$$D\Big(\sum_g A_g \;\Big\|\; \sum_g B_g\Big) = \sum_g D(A_g\|B_g). \tag{17}$$

[13]Formal declarations: `Matrix.rawSupportSourceAQuadratic_sum_sub_nonneg` and `Matrix.supportSourceBQuadratic_sum_sub_nonneg`, `Matrix.supportLeftRightSupportInv_mulVec_sourceB_eq_of_defect_eq_zero`, and `Matrix.supportRightProj_mul_sumSupportRightProj_eq` in `TNLean/Channel/Schwarz/SupportSourceDefect.lean`.

[14]Formal declarations: `Matrix.supportRelativeEntropyFamilyGapIntegrand`, `Matrix.supportRelativeEntropyFamilyGapIntegrand_integrableOn_and_integral_eq`, and `Matrix.supportRelativeEntropyFamilyGapIntegrand_continuousOn` in `TNLean/Channel/Schwarz/SupportRelativeEntropyGap.lean`.

[15]Formal declarations: `Matrix.supportRelativeEntropyFamilyGapIntegrand_nonneg` and `Matrix.supportSourceBDefect_eq_zero_of_relativeEntropy_sum_eq` in `TNLean/Channel/Schwarz/SupportRelativeEntropyGap.lean`.

[16]Formal declarations: `Matrix.supportRightProj_mul_supportLeftRightSupportProj_eq` in `TNLean/Channel/Schwarz/SupportLeftRightRelativeModular.lean`, and `Matrix.supportRelativeModular_resolvent_mulVec_eq_of_relativeEntropy_sum_eq` in `TNLean/Channel/Schwarz/SupportRelativeEntropyEquality.lean`.

Unitary conjugation transports the local kernel inclusions, while positivity of the summands transports them to the summed pair. Hence the generic theorem gives the common projected relative-modular resolvent for every Weyl summand. At the zero Weyl index, the local pair is exactly (ρ, σ). [17] This is the analytic specialization required before the support square-root-ratio step. It does not yet assert that ratio, the Petz sandwich, or recovery. Following this step, consider two positive semidefinite pairs (A, B) and (C, D). Let P_B be the support projection of the first reference matrix B, and write $B^+ = (B_{\mathrm{supp}}^{-1/2})^2$ and $D^+ = (D_{\mathrm{supp}}^{-1/2})^2$. Suppose that, for every $t > 0$, their shifted generalized relative-modular resolvents satisfy

$$\begin{aligned}(\mathbf{1} \otimes P_B^{\mathsf{T}})\big(t\mathbf{1} + A \otimes (B^+)^{\mathsf{T}}\big)^{-1} \operatorname{vec}(\mathbf{1}^{\mathsf{T}}) \\ = (\mathbf{1} \otimes P_B^{\mathsf{T}})\big(t\mathbf{1} + C \otimes (D^+)^{\mathsf{T}}\big)^{-1} \operatorname{vec}(\mathbf{1}^{\mathsf{T}}).\end{aligned} \tag{18}$$

Then their square-root ratios agree on the same support:

$$\big(\sqrt{A}\, B_{\mathrm{supp}}^{-1/2}\big) P_B = \big(\sqrt{C}\, D_{\mathrm{supp}}^{-1/2}\big) P_B. \tag{19}$$

The orientation of the vectorized restriction is fixed by

$$(\mathbf{1} \otimes P_B^{\mathsf{T}}) \operatorname{vec}(M^{\mathsf{T}}) = \operatorname{vec}\big((M P_B)^{\mathsf{T}}\big). \tag{20}$$

Here the generalized inverse is the square of the support inverse square root. [18] The restriction is essential. Equality on the ambient vectorized identity would force equality of the two reference support projections, since each resolvent contributes t^{-1} on the complement of its reference support. This is false in the intended data-processing application: if $\rho = \sigma$ is a rank-one maximally entangled state, equality under partial trace is automatic, whereas the comparison pair formed from its marginal is full rank. Thus the ambient resolvent equality is not a consequence of the singular equality theorem.

The canonical source-side projected resolvent is known to solve its singular left–right equation: after projection by $\mathbf{1} \otimes P_B^{\mathsf{T}}$, multiplication by

[17] Formal declarations: `Matrix.quantumRelativeEntropy_smul_support` in `TNLean/Analysis/RelativeEntropySupportIntegral.lean`, and `Matrix.weyl_supportRelativeModular_resolvent_mulVec_eq_of_partialTraceRight_eq` in `TNLean/Channel/Schwarz/WeylSupportRelativeEntropyEquality.lean`.

[18] Formal declarations: `Matrix.one_kronecker_transpose_mulVec_vec_transpose`, `Matrix.supportRelativeModular_sourceB_solution`, `Matrix.supportRelativeModular_sqrt_mulVec_vec_one` and `Matrix.supportRelativeModular_sqrt_ratio_eq_of_resolvent_mulVec_eq` in `TNLean/Channel/Schwarz/SupportRelativeModular.lean`.

$A\otimes\mathbf{1}+t(\mathbf{1}\otimes B^{\mathsf{T}})$ gives $\mathrm{vec}(B^{\mathsf{T}})$. This uses only the generalized-inverse identity $(B_{\mathrm{supp}}^{-1/2})^2 B = P_B$. The one-pair identity above also identifies this vector with the support-generalized-inverse solution of the left–right equation, and hence identifies the two source-B quadratic forms. Together with the integral and continuity argument above, equality of relative entropies now makes the real-valued source-B defect vanish at every positive parameter. Nonnegativity of the complex quadratic residuals makes their imaginary parts vanish, so the real equality yields the finite-family common projected resolvent theorem. The kernel inclusion used for the unprojected source-A inequality remains an exact hypothesis of this source-facing equality theorem. The finite-Weyl specialization now supplies the projected resolvent hypothesis for the support functional-calculus step. The zero Weyl summand, the unweighted-sum normalization, and the projected square-root-ratio theorem now carry this conclusion through the singular inverse-square-root sandwich and raw Petz-recovery argument. [19]

In particular, the affine regularizations used to prove the inequality cannot by themselves prove its equality case. Equality at the limiting pair implies only that the nonnegative regularized gaps tend to zero; it does not imply that any gap at positive regularization parameter is zero. Thus the positive-definite equality theorem cannot be applied pointwise to those regularized pairs without an additional equality-preservation theorem.

There is nevertheless a useful algebraic reduction for the identity summand. Write

$$\overline{X} = (\mathrm{tr}_R\, X)\otimes d_R^{-1}\mathbf{1}_R.$$

For every positive semidefinite τ on the left factor,

$$(\tau\otimes d_R^{-1}\mathbf{1}_R)_{\mathrm{supp}}^{-1/2} = \sqrt{d_R}\,(\tau_{\mathrm{supp}}^{-1/2}\otimes\mathbf{1}_R), \tag{21}$$

$$\overline{\sigma}_{\mathrm{supp}}^{-1/2}\,\overline{\rho}\,\overline{\sigma}_{\mathrm{supp}}^{-1/2} = \big((\mathrm{tr}_R\,\sigma)_{\mathrm{supp}}^{-1/2}(\mathrm{tr}_R\,\rho)(\mathrm{tr}_R\,\sigma)_{\mathrm{supp}}^{-1/2}\big)\otimes\mathbf{1}_R. \tag{22}$$

The second identity follows because the two factors $\sqrt{d_R}$ cancel the coefficient d_R^{-1} in $\overline{\rho}$. Hence the operator identity

$$\sqrt{\sigma}\,\overline{\sigma}_{\mathrm{supp}}^{-1/2}\,\overline{\rho}\,\overline{\sigma}_{\mathrm{supp}}^{-1/2}\,\sqrt{\sigma} = \rho$$

[19]Formal declarations: `Matrix.unweightedWeylSum_eq_smul_partialTraceRight_kronecker` in `TNLean/Channel/Schwarz/WeylSupportRelativeEntropyEquality.lean`; `Matrix.weyl_support_sqrt_ratio_eq_of_partialTraceRight_eq`, `Matrix.weyl_support_identity_sandwich_of_partialTraceRight_eq`, and `Matrix.partialTraceRightPetzMap_eq_of_relativeEntropy_eq` in `TNLean/Channel/Schwarz/WeylSupportPetzRecovery.lean`.

implies $\mathcal{R}_\sigma(\operatorname{tr}_R \rho) = \rho$ by the support Petz formula of [2, Theorem 3, equation (8)]. [20] The algebraic implication itself is conditional on the displayed operator identity. The finite-Weyl argument now supplies that identity from equality of relative entropy under the partial trace for positive-semidefinite inputs satisfying $\ker \sigma \subseteq \ker \rho$. No claim is made here for an arbitrary convex ensemble.

The singular-support equality case of data processing is first formalized in finite Weyl coordinates:

$$D(\rho\|\sigma) = D(\operatorname{tr}_R \rho \| \operatorname{tr}_R \sigma) \quad \Longrightarrow \quad \mathcal{R}_\sigma(\operatorname{tr}_R \rho) = \rho. \tag{23}$$

This is the raw support-map identity entering the forward implication of [2, Theorem 3]. The proof does not infer equality for positive affine regularizations. Instead, the support-restricted finite-family equality theorem gives the common projected Weyl resolvent; the support functional calculus gives the projected square-root ratio; and the kernel inclusion reduces its Gram identity to the displayed Petz sandwich.

Product-coordinate covariance transports this identity to arbitrary finite tensor factors. For density operators, the support condition also makes the chosen trace-preserving completion agree with the raw support formula on $\operatorname{tr}_R \rho$. Consequently the source-facing conclusion is

$$D(\rho\|\sigma) = D(\operatorname{tr}_R \rho \| \operatorname{tr}_R \sigma) \quad \Longrightarrow \quad \widehat{\mathcal{R}}_\sigma(\operatorname{tr}_R \rho) = \rho. \tag{24}$$

The complementary preparation term in $\widehat{\mathcal{R}}_\sigma$ is the chosen trace-preserving extension and is not part of [2, Equation (8)]. [21]

For the exact strong-subadditivity pair $\rho = \rho_{ABC}$ and $\sigma = \rho_A \otimes \rho_{BC}$, the product-marginal support inclusion is automatic. The specialized declarations

```
Matrix.product_marginal_support
Matrix.partialTraceRightPetzMap_product_marginal_recovery_of_isSSAEq
uality
Matrix.partialTraceRightPetzChannel_traceA_ABC_isKrausCPTP
Matrix.idTensor_partialTraceRightPetzChannel_traceC_ABC_eq_of_isSSAE
quality
```

[20] Formal declarations: `Matrix.PosSemidef.supportInvSqrt_kronecker_maximallyMixed`, `Matrix.PosSemidef.supportInvSqrt_kronecker_maximallyMixed_mul_mul`, and `Matrix.partialTraceRightPetzMap_eq_of_weyl_identity_sandwich` in `TNLean/Channel/Schwarz/PetzEqualityPrerequisites.lean`.

[21] Formal declarations: `Matrix.partialTraceRightPetzMap_submatrix_prod_equiv`, `Matrix.partialTraceRightPetzMap_eq_of_relativeEntropy_eq_general_support`, and `Matrix.partialTraceRightPetzChannel_recovery_of_dpi_equality`.

therefore give HJPW equation (11):

$$\rho_{ABC} = (\mathrm{id}_A \otimes \widehat{\mathcal{R}}_{\rho_{BC}})(\rho_{AB}). \tag{25}$$

The local completed Petz map is trace-preserving and completely positive. For singular ρ_A, this is a supported-input identity, not a global factorization of TNLean's generic completed product-reference channel.

The full-support invariant-state decomposition and preserving-operation block action used in [2, Theorem 6] are formalized, as are the joint-support reduction and its application to the invariant conditional family. The declarations

```
Matrix.exists_markovBipartiteBlockForm_jointSupport
Matrix.exists_ambientMarkovBipartiteBlockForm
Matrix.exists_markovDilationBlockForm
Matrix.MarkovDilationBlockForm.ambient_tripartite_block_form
Matrix.exists_markovTripartiteBlockForm
```

produce the positive unnormalized factors ω_j, prove that their traces sum to one, produce the trace-one supported sector output states $\widehat{\rho}_j$, and establish (2) with zero complementary blocks. The theorem `Matrix.hayashi_ssa_equality_characterization_forward` then sets $p_j = \mathrm{Re}\,\mathrm{tr}\,\omega_j$, reverses the fibre order, uses the adjoint middle-system unitary, and performs the final total normalization. Every supported right factor remains its sector output state, including in a zero-weight supported sector. A zero-weight supported sector may use a filler only on the left, whereas a complementary sector may use fillers on both sides.

The singular-support recovery scope is unchanged. Equality in data processing gives the raw support-Petz recovery identity, and on the Petz output marginal the chosen completed local channel agrees with that raw map. No tensor-product factorization is claimed for the generic completed singular product-reference channel away from the supported input.

4 Verdict

Closed. The forward and reverse implications of the strong-subadditivity equality characterization are proved, and their biconditional is axiom-free. The forward proof derives the ambient HJPW blocks, uses $p_j = \mathrm{Re}\,\mathrm{tr}\,\omega_j$ with nonnegative weights summing to one, preserves the trace-one sector output states on supported sectors, and introduces arbitrary normalized fillers only where the final weight is zero. The middle fibres are read in $B_j^L \otimes B_j^R$ order, and the Hayashi unitary has the adjoint orientation required by (2).

The closure does not strengthen the completed Petz map off support. The proved source-facing claim remains the raw support-map recovery identity, together with agreement of the chosen completed local channel on the supported marginal.

References

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[2] P. Hayden, R. Jozsa, D. Petz, and A. Winter, *Structure of states which satisfy strong subadditivity of quantum entropy with equality*, Communications in Mathematical Physics **246** (2004), 359–374.

[3] M. Hayashi, *Quantum Information: An Introduction*, Springer, 2006; Theorem 5.24.