Nilpotent Basis Elements in the Simple-Tensor Definition of Cirac–Pérez-García–Schuch–Verstraete

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Verdict: local-correction (resolved).

1 Source definition

Cirac–Pérez-García–Schuch–Verstraete define, at arXiv:1606.00608, line 822: "A tensor generating MPDO's is simple if none of the elements in a BNT is nilpotent." The meaning of nilpotency is fixed at line 819: an element $\mathcal{B}_k$ of the basis of normal tensors is nilpotent when $\operatorname{tr}_R \rho^{(N)}(M_k) = 0$ after tracing $R \leq D$ sites, for all sufficiently large N.

2 Formal reading

Let $A = \mathcal{B}_k$ be a normal BNT representative and put $\mathcal{T}_A = \sum_i A^{ii}$. Here $\operatorname{tr}_R \rho^{(N)}(A) = 0$ means that the reduced density *operator* is zero after tracing R consecutive sites. Its matrix entries are

$$\left(\operatorname{tr}_R \rho^{(N)}(A)\right)_{\mathbf{i},\mathbf{j}} = \operatorname{tr}\left(A^{i_1 j_1} \cdots A^{i_{N-R} j_{N-R}} \mathcal{T}_A^R\right). \tag{1}$$

Choose a sufficiently large N for which the length-$(N-R)$ products of the normal representative span the full virtual matrix algebra. The trace pairing on that algebra is nondegenerate. Consequently,

$$\operatorname{tr}_R \rho^{(N)}(A) = 0 \text{ for every sufficiently large } N \quad \Longleftrightarrow \quad \mathcal{T}_A^R = 0. \tag{2}$$

Since the nilpotency index of a nilpotent $D \times D$ matrix is at most D, the line-819 condition is precisely nilpotency of $\mathcal{T}_A$. The documented active canonical-block reading is `MPOTensor.IsSourceSimple`: it existentially quantifies over the positive physical blocking required at line 815 and chooses an active BNT sector presentation of the blocked doubled-index tensor. Every representative has a positive number of copies, every copy has nonzero weight, the representatives are normal and eventually linearly independent, and the blocked tensor has the same positive-length matrix product vectors as their weighted direct sum. The predicate records non-nilpotency for every active representative. Thus dormant candidates allowed by the raw coefficient-form predicate `MPSTensor.IsCPSVBasisOfNormalTensors` do not enter this corrected reading. The nonempty representative family retains the nonzero-family boundary from the line-246 convention without imposing its unit-modulus normalization.

This is a local correction, not a consequence of the literal BNT definition. That definition permits adjoining a normal tensor with coefficient identically zero, and the resulting failure of unrestricted BNT uniqueness is documented in `docs/paper-gaps/cpsv16_bnt_uniqueness_zero_coefficient.tex`.

Positive-length nontriviality is a theorem rather than a defining assumption. Indeed, a nonzero copy-weight power sum cannot vanish eventually; at a large length where one such coefficient is nonzero, eventual linear independence makes the blocked matrix product vector nonzero. Physical unblocking then gives `MPOTensor.IsSourceSimple.exists_mpo_ne_zero`. Isolated vanishing lengths remain possible. The strengthened project predicate `MPOTensor.IsNonvanishingSourceSimple` additionally requires the generated MPO to be nonzero at every positive chain length; that clause is not part of Definition 4.7.

The predicates `MPOTensor.IsSimpleCanonicalForm` and `MPOTensor.IsSimple` are separate normalized fixed-representative predicates. They include a normalized sector decomposition with the source's line-246 unit-weight convention. No equivalence with `MPOTensor.IsSourceSimple` is asserted.

The toric-code boundary tensor discussed after Theorem 4.9 of the source confirms the reading: its second basis element carries the one-by-one transfer $\mathcal{T} = 0$, so the tensor is not simple, matching the source's use of that example outside the simple case.

The source states the predicate for "a BNT" of the blocked tensor and then asserts that BNTs are unique up to gauge, phase, and permutation. The latter assertion is false for the literal definition: a dormant normal tensor can be adjoined with coefficient zero at every positive length. Within the restricted class of active presentations, the theorem `MPSTensor.IsActiveCPSVBasisOfNormalTensors.equiv_of_sameMPVPos` matches two active presentations, and `MPOTensor.activeBNT_basis_not_isNilpotent_iff` transports nilpotency through the resulting permutation, dimension identification, gauge, and phase. Thus the choice is immaterial only after the active qualification. The active predicate does not require a normalized horizontal canonical-form presentation.

3 Classification

Verdict: the formal predicate is complete for the documented active canonical-block reading of lines 217–246 and Definition 4.7, under the identification of the line-819 partial-trace criterion with matrix nilpotency of the physical-trace transfer. It is not the unrestricted literal-BNT reading, whose presentation independence is false. The predicate requires an active, nonempty BNT sector presentation; positive-length nontriviality follows from that presentation. It does not require nonvanishing at every positive length. The latter is available only in the separate strengthened project predicate. The previous raw coefficient-form witness, together with a separately assumed nonzero MPO, was definition drift because it could still contain dormant candidates.