Closure record for saturation of the area law from MPDO renormalization fixed points

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Verdict: clarification (resolved).

1 Source statement and normalization boundary

Definition 4.1 of Cirac–Pérez-García–Schuch–Verstraete defines a mixed-state renormalization fixed point by trace-preserving completely positive maps $\mathcal{S}$ and $\mathcal{T}$ satisfying[1]

$$\mathcal{S}[M_2(X)] = M_1(X), \qquad \mathcal{T}[M_1(X)] = M_2(X)$$

for every virtual boundary operator X. Proposition `propsimple` in Appendix C states that this condition implies source zero correlation length and saturation of the area law.[2]

The entropic convention immediately preceding Definition 4.6 divides each finite-chain operator by its trace.[3] The precise boundary of the proposition is therefore:

1. $\rho^{(N)}(M)$ is positive semidefinite for every $N \geq 1$;
2. $\operatorname{tr} \rho^{(N)}(M) \neq 0$ for every $N \geq 1$;
3. M satisfies the local renormalization equations of Definition 4.1.

Positivity alone does not imply the second condition: the zero tensor generates zero positive semidefinite operators at every length. Thus the nonzero-trace clause is the normalization boundary implicit in the source's density-operator language, not a dispensable consequence of positivity.

[1] Source label `RFPMixedTS`, `Papers/1606.00608/MPDO-22-12-17-2.tex:638--660`.
[2] Source: `Papers/1606.00608/MPDO-22-12-17-2.tex:1331--1341`.
[3] Source: `Papers/1606.00608/MPDO-22-12-17-2.tex:792--793`.

2 Area-law calculation

For a bipartition of an N-site ring into consecutive intervals of lengths $a+1$ and $b+2$, the localized maps satisfy

$$(\widehat{\mathcal{T}}_a \otimes \widehat{\mathcal{S}}_b)(\sigma^{(N)}_{a+1:b+2}) = \sigma^{(N)}_{a+2:b+1},$$

and the reverse pair satisfies

$$(\widehat{\mathcal{S}}_a \otimes \widehat{\mathcal{T}}_b)(\sigma^{(N)}_{a+2:b+1}) = \sigma^{(N)}_{a+1:b+2}.$$

These are identities of normalized density operators because the corresponding ring trace is nonzero. Mutual-information data processing in the two directions gives

$$I(\sigma^{(N)}_{a+2:b+1}) \le I(\sigma^{(N)}_{a+1:b+2}) \le I(\sigma^{(N)}_{a+2:b+1}).$$

Hence the two mutual informations are equal. The consecutive-cut identification turns this equality into $I_L = I_{L+1}$ for $1 \le L < \lfloor N/2 \rfloor$.

The primary formal declaration is `MPOTensor.isSAL_of_isRFPViaTS_of_trace_ne_zero` in `TNLean/MPS/MPDO/RFPViaTSSAL`. The channel identities and the bipartite-to-chain identification are supplied by `MPOTensor.tensorMapBoth_bipartitionedNormalizedMPO`, `MPOTensor.tensorMapBoth_bipartitionedNormalizedMPO_reverse`, and `MPOTensor.mutualInfoChain_eq_mutualInformation`.

3 Source zero correlation length

Trace preservation in the one-to-two renormalization equation gives, for every virtual matrix X,

$$\operatorname{tr}(\mathcal{T}_M^2 X) = \operatorname{tr}(M_2(X)) = \operatorname{tr}(M_1(X)) = \operatorname{tr}(\mathcal{T}_M X).$$

Nondegeneracy of the trace pairing yields $\mathcal{T}_M^2 = \mathcal{T}_M$. If $\mathcal{T}_M = 0$, then

$$\operatorname{tr} \rho^{(1)}(M) = \operatorname{tr}(\mathcal{T}_M) = 0,$$

contradicting the positive-length nonvanishing hypothesis. Therefore $\mathcal{T}_M \neq 0$, and M has source zero correlation length.

The literal Definition 4.2 equation is the direct theorem `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS` in `TNLean/MPS/MPDO/RFPViaTS`. Combining it with `MPOTensor.isSAL_of_isRFPViaTS_of_trace_ne_zero` gives exactly the source-facing forward implication: positive semidefinite rings, nonzero trace at every positive length, and the Definition 4.1 local renormalization equations imply literal ZCL and SAL. The convenience theorem `MPOTensor.isSourceZCL_and_isSAL_of_isRFPViaTS_of_trace_ne_zero` records the broader scale-invariant ZCL corollary and is not used as the paper-facing statement.

4 Horizontal specialization

The declaration `MPOTensor.isSAL_of_isRFPViaTS` is the horizontal SAL specialization. Together with the same literal theorem `MPOTensor.physTraceTransfer_sq_of_isRFPViaTS`, it yields the paper-facing horizontal corollary. The convenience declaration `MPOTensor.isSourceZCL_and_isSAL_of_isRFPViaTS` packages the broader scale-invariant conclusion. The normalized BNT-refined horizontal hypothesis is stronger than the density-family boundary above. It supplies

$$\operatorname{tr} \rho^{(N)}(M) \neq 0$$

for every $N \geq 1$ through `MPOTensor.trace_mpo_ne_zero_of_isHorizontalCF_isMPDO_isRFPViaTS`. No horizontal decomposition is used in the mutual-information calculation itself.

The literal source canonical-form construction permits an additional all-zero virtual summand because it allows $\sum_k D_k \leq D$.[4] That observation no longer leaves Proposition `propsimple` open: the proposition is stated and proved at its actual density-operator normalization boundary, while horizontal form gives a stronger sufficient condition for that boundary.

Classification: boundary condition. Proposition `propsimple` is complete at the explicit nonzero-trace density-family boundary. Theorem 4.9 is partial and partly refuted for independent reasons. The normal Case-I form of Lemma C.5 is complete, while the Case-II passage after coefficient absorption remains a missing statement with proof-path drift. Separately, unequal raw repeated-copy weights refute the literal implication (iv)⇒(v). The viable source (ii)⇒(v) route remains open after ZCL-derived common-weight absorption and projector-controlled channel assembly. No simplicity hypothesis is used in the completed forward implication.

[4]Source: `Papers/1606.00608/MPDO-22-12-17-2.tex:214--219`.