# Per-block RFP Isometry Form and the Source Isometry Condition in CPSV16

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**Verdict:** false-source (open).

## 1 Source statement

The pure-state RFP characterization in Section 3 of [CPGSV16] writes a canonical-form RFP tensor as

$$A^i = \bigoplus_{j=1}^{g} \bigoplus_{q=1}^{r_j} \mu_{j,q} X_{j,q} \Lambda_j U_j^i X_{j,q}^{-1}, \tag{III\_CFI\_RFP}$$

where the matrices $\Lambda_j$ are diagonal, positive, and trace-normalized. The same theorem requires a joint isometry condition

$$\sum_{i=1}^{s} (U_j^i)_{\alpha,\beta} \overline{(U_{j'}^i)_{\alpha',\beta'}} = \delta_{j,j'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \tag{eq:III\_isometry}$$

The diagonal case $j = j'$ is a full pair-index orthonormality condition in the two virtual indices $(\alpha, \beta)$. The factor $\delta_{j,j'}$ adds a cross-block orthogonality condition: for $j \neq j'$, the physical-index isometries associated with the two normal-tensor blocks are mutually orthogonal. Corollary III.cor3 invokes this same isometry condition for the BNT elements.[1]

## 2 Formal boundary

The single-normal-tensor characterization is formalized under the exact source hypotheses. If $A$ is normal in the sense of CPSV16, then its bond dimension is necessarily positive by `MPSTensor.IsNormalTensor.bondDim_ne_zero`, and one has

$$A \text{ is a renormalization fixed point} \iff A \text{ is in the square-root isometry canonical form.} \tag{1}$$

[1] Local source: `Papers/1606.00608/MPDO-22-12-17-2.tex:543--554` and `Papers/1606.00608/MPDO-22-12-17-2.tex:583--589`.

The forward implication passes to the trace-preserving Perron gauge, applies the left-canonical structural theorem there, and returns along the similarity; the reverse implication follows directly from the rank-one transfer formula.[2]

The earlier per-block theorem remains useful for an indexed family.[3] For each block $k$ it gives

$$A_k^i = X_k \Lambda_k U_k^i X_k^{-1}, \tag{2}$$

$$\sum_i (U_k^i)^\dagger U_k^i = I, \tag{3}$$

$$\sum_i \overline{(U_k^i)_{\alpha,\beta}}\,(U_k^i)_{\alpha',\beta'} = D_k^{-1}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{4}$$

The second equation is the contracted identity

$$\sum_{i,\alpha} \overline{(U_k^i)_{\alpha,\beta}}\,(U_k^i)_{\alpha,\beta'} = \delta_{\beta,\beta'}. \tag{5}$$

The third equation records the diagonal pair-index orthonormality in the normalization used by the formal theorem. Compared with the source convention, the entries of $U_k^i$ carry the normalized matrix-unit factor $D_k^{-1/2}$; therefore the compatible pair-index equation is

$$\sum_i \overline{(U_k^i)_{\alpha,\beta}}\,(U_k^i)_{\alpha',\beta'} = D_k^{-1}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{6}$$

The remaining normalization comparison with the source equation `eq:III_isometry` is therefore explicit rather than hidden.

A subsequent comparison theorem rescales the same single-block decomposition by $U_k^i \mapsto D_k^{1/2} U_k^i$ and $\Lambda_k \mapsto D_k^{-1/2}\Lambda_k$.[4] This leaves the product $\Lambda_k U_k^i$ unchanged and gives the diagonal pair-index equation in the source unit convention

$$\sum_i \overline{(U_k^i)_{\alpha,\beta}}\,(U_k^i)_{\alpha',\beta'} = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{7}$$

Thus the scalar normalization mismatch is no longer hidden in the formal statement: both conventions are available, and their comparison is explicit. The Appendix B structural datum used in the conditional parent-Hamiltonian theorem chooses the unit pair-index convention, so the remaining coefficient problem starts from the source normalization. Appendix B first gives the coefficient form

[2] Formal declarations: `MPSTensor.GaugeEquiv.isIsometryCanonicalForm_iff` in `TNLean/MPS/RFP/StructuralFull` and `MPSTensor.IsNormalTensor.isTransferIdempotent_iff_isIsometryCanonicalForm` in `TNLean/MPS/RFP/NormalIsometryCharacterization`.

[3] Formal declaration: `MPSTensor.rfp_nt_structural_full_blocks` in `TNLean/MPS/RFP/StructuralFull`. Introduced in PR #2785.

[4] Formal declarations: `MPSTensor.rfp_nt_structural_full_unit_pair` and `MPSTensor.rfp_nt_structural_full_blocks_unit_pair` in `TNLean/MPS/RFP/StructuralFull`.

of the basic-vector expression

$$C_L(\sigma) = \sum_{\alpha_0,\dots,\alpha_{L-1}} \prod_{t=0}^{L-1} \Lambda_{\alpha_t} (U^{\sigma_t})_{\alpha_t,\alpha_{t+1}}, \qquad \alpha_L = \alpha_0, \tag{8}$$

which is the coefficient form of $|V^{(L)}(A_j)\rangle = U^{\otimes L}|\varphi_j\rangle^{\otimes L}$, with $\varphi_j$ shared between $b_t$ and $a_{t+1}$. The even-chain physical-pair factorization considered in the conditional parent-Hamiltonian theorem is therefore a separate statement, not the source expression itself. The formal development proves this cyclic coefficient formula for the original tensor after the Appendix B change of basis, for every nonempty chain length and a chosen structural datum:

$$V^{(L)}(A)_\sigma = C_L(\sigma).$$

This is the declaration `MPSTensor.AppendixBStructuralData.mpv_eq_cyclicVirtualPairState`. It does not prove the even-chain physical-pair factorization. The pair-index isometry above is an orthogonality equation after summing over the physical letter $i$. It is therefore not, by itself, a coefficient identity for a fixed physical word $\sigma$. Any proof of a fixed-word physical-pair factorization must use the source basic-vector expression $U^{\otimes L}\varphi_j^{\otimes L}$, or an equivalent construction of the corresponding local projectors, rather than applying the summed isometry as if the physical labels were also summed in $C_L(\sigma)$. This obstruction is genuine, not merely a missing formal proof. For example, take two virtual indices and physical labels $i = (a, b)$, with

$$(U^{(a,b)})_{x,y} = \delta_{x,a}\delta_{y,b}, \qquad A^{(a,b)} = \lambda_a E_{a,b},$$

where $\lambda_0, \lambda_1 > 0$. The unit pair-index isometry holds. For the length-four word

$$\sigma = ((0, 1), (1, 0), (1, 0), (0, 1)),$$

the cyclic coefficient $C_4(\sigma)$ vanishes, because the middle virtual matching condition fails. The physical-pair product is instead

$$(\lambda_0\lambda_1)(\lambda_1\lambda_0) = \lambda_0^2\lambda_1^2 \neq 0.$$

Thus the even-chain physical-pair coefficient identity is not a consequence of the unit pair-index isometry alone.

The per-block isometry form does not by itself relate $U_k$ and $U_\ell$ for $k \neq \ell$. The cross-block equations

$$\sum_i (U_k^i)_{\alpha,\beta}\, \overline{(U_\ell^i)_{\alpha',\beta'}} = 0 \qquad (k \neq \ell), \tag{9}$$

the $j \neq j'$ part of the source condition `eq:III_isometry`, are established separately at the normal-tensor-block level, in the section on the cross-block residual isometry below.

## 3 Square-root diagonal (local correction)

The source's displayed normal-tensor form (`eq_III_NT_RFP`) is written

$$A^i = X\,\Lambda\,U^i\,X^{-1}, \qquad (\texttt{eq\_III\_NT\_RFP})$$

with the diagonal $\Lambda$ itself, together with the unit isometry condition[5]

$$\sum_i (U^i)_{\alpha,\beta}\,\overline{(U^i)_{\alpha',\beta'}} = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{10}$$

This literal pairing is internally inconsistent. The same proof constructs the reference tensor[6]

$$\widehat{A}^{(\alpha',\beta')}_{\alpha,\beta} = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\sqrt{\Lambda_\alpha}, \tag{11}$$

which is left-canonical and has fixed point $\operatorname{diag}(\Lambda)$ with $\operatorname{tr}(\Lambda) = 1$. Writing $\widehat{A} = \Lambda\,U$ would force $U^{(\alpha',\beta')}_{\alpha,\beta} = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}/\sqrt{\Lambda_\alpha}$, whose pair-index sum is

$$\sum_i (U^i)_{\alpha,\beta}\,\overline{(U^i)_{\alpha',\beta'}} = \frac{\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}}{\Lambda_\alpha} \neq \delta_{\alpha,\alpha'}\delta_{\beta,\beta'} \qquad (\Lambda_\alpha \neq 1), \tag{12}$$

so the residual after factoring $\Lambda$ is *not* a unit isometry. Writing instead $\widehat{A} = \sqrt{\Lambda}\,U$ gives $U^{(\alpha',\beta')}_{\alpha,\beta} = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$, whose pair-index sum is $\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$, a genuine unit isometry.

The formalization therefore states the decomposition with the square-root diagonal,

$$A^i = X\,\sqrt{\Lambda}\,U^i\,X^{-1}, \tag{13}$$

keeping both the source trace-normalization $\operatorname{tr}(\Lambda) = 1$ on $\Lambda$ and the source unit isometry condition on $U$ simultaneously. This $\Lambda \to \sqrt{\Lambda}$ change is a *local correction* of the source display: it is mathematically correct as stated, and the source itself uses $\sqrt{\Lambda_\alpha}$ for the reference tensor at line 1300, so the displayed $\Lambda$ at line 1278 is a typographic simplification of the dressing.

## 4 Trace normalization (discharged)

A named predicate records the trace-normalized isometry canonical form.[7] The predicate asserts the existence of an invertible $X$, a positive diagonal $\Lambda$ with $\sum_\alpha \Lambda_\alpha = 1$, and a unit pair-index isometry $U$ with

$$A^i = X\,\sqrt{\Lambda}\,U^i\,X^{-1}, \qquad \sum_i \overline{(U^i)_{\alpha,\beta}}\,(U^i)_{\alpha',\beta'} = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{14}$$

[5] Local source: `Papers/1606.00608/MPDO-22-12-17-2.tex:1278` for the displayed equation and `Papers/1606.00608/MPDO-22-12-17-2.tex:1281-1283` for the unit isometry.

[6] Local source: `Papers/1606.00608/MPDO-22-12-17-2.tex:1300`.

[7] Formal declaration: `MPSTensor.IsIsometryCanonicalForm` in `TNLean/MPS/RFP/StructuralFull`.

The square root on the diagonal matches the source reference tensor (the square-root correction is documented separately in the previous section); the trace condition $\mathrm{tr}(\Lambda) = 1$ is imposed on $\Lambda$ itself, equivalently on the normalized diagonal fixed point $\rho = \mathrm{diag}(\Lambda)$, $\Lambda_\alpha = \rho_{\alpha,\alpha}/\operatorname{tr}\rho$. The identity $\sum_\alpha \Lambda_\alpha = 1$ is the trace identity $\sum_\alpha \rho_{\alpha,\alpha} = \operatorname{tr}\rho$ for the diagonal fixed point.

The left-canonical forward theorem `MPSTensor.isIsometryCanonicalForm_of_rfp_nt` proves this form for the Perron representative. Gauge invariance then proves that every normal tensor is a renormalization fixed point if and only if it admits the displayed square-root form, with no bond-dimension or left-canonical hypothesis on the original tensor. This is `MPSTensor.IsNormalTensor.isTransferIdempotent_iff_isIsometryCanonicalForm`.

This discharges the single-normal-tensor statement and its trace-normalization scope item. The cross-block $\delta_{j,j'}$ orthogonality between distinct normal-tensor blocks is treated in the next section.

# 5 Cross-block residual isometry

The cross-block $\delta_{j,j'}$ orthogonality is formalized at the level of the normal-tensor blocks. For a family of normal, irreducible, left-canonical blocks $(A_j)_j$, no two gauge-phase equivalent, with $\dim_j \geq 1$ and whose direct sum $\bigoplus_j A_j$ is a renormalization fixed point, the mixed transfer operators between distinct blocks vanish,

$$\mathcal{E}_{j,j'} = \sum_i A_j^i \otimes \overline{A_{j'}^i} = 0 \qquad (j \neq j'),$$

formalized by `MPSTensor.isBNTLocallyOrthogonal_of_isTransferIdempotent_directSum`. The argument decomposes the direct-sum transfer map block by block (`MPSTensor.blockDiagonal'_transferSum_toBlock`): whole-tensor idempotence makes each $\mathcal{E}_{j,j'}$ idempotent, and the same-dimension and dimension-mismatch transfer-operator gaps force an idempotent of spectral radius below one to be zero.

Passing to the residual isometries uses the isometry canonical form of each block. Writing $A_j^i = X_j \sqrt{\Lambda_j}\, U_j^i X_j^{-1}$ with $X_j$ and $\sqrt{\Lambda_j}$ invertible, the mixed transfer operator is covariant under the two-sided change of variables,

$$\mathcal{E}_{A_j,A_{j'}}(Y) = X_j \sqrt{\Lambda_j}\, \mathcal{E}_{U_j,U_{j'}}\big(X_j^{-1} Y (X_{j'}^{-1})^\dagger\big)\, (X_{j'} \sqrt{\Lambda_{j'}})^\dagger,$$

formalized by `MPSTensor.mixedTransferMap_conj_apply`. Cancelling the invertible outer factors converts $\mathcal{E}_{A_j,A_{j'}} = 0$ into $\mathcal{E}_{U_j,U_{j'}} = 0$, and reading the matrix entry at a matrix unit yields the residual-isometry equation $\sum_i (U_j^i)_{\alpha,\beta} \overline{(U_{j'}^i)_{\alpha',\beta'}} = 0$. Together with the within-block orthonormality of the isometry canonical form, this is the full source isometry condition `eq:III_isometry`, packaged as the predicate `MPSTensor.IsResidualIsometryFamily` and established under whole-tensor renormalization fixed point of the direct sum by `MPSTensor.exists_residualIsometryFamily_of_isTransferIdempotent_directSum`. This is the residual-isometry content of Corollary III.cor3 at the normal-tensor-block level.

# 6 Corollary 3.12: arbitrary and active BNT members

The printed Corollary 3.12 applies the joint residual-isometry conclusion to an arbitrary BNT of a canonical-form renormalization fixed point. This is false because the BNT definition permits a listed normal tensor whose coefficient vanishes at every positive length. Such a tensor need not be orthogonal to the active members.

A bond-one example has two physical letters and

$$A^0 = B_0^0 = 1, \qquad A^1 = B_0^1 = 0, \qquad B_1^0 = B_1^1 = \frac{1}{\sqrt{2}}. \tag{15}$$

The tensor $A$ is in CPSV canonical form and has identity transfer map. The family $(B_0, B_1)$ is eventually linearly independent and spans every positive-length matrix product vector of $A$ with coefficients 1 and 0, respectively. It is therefore a BNT for $A$. Both $B_0$ and $B_1$ have identity transfer map, but

$$\sum_{i=0}^{1} B_0^i \overline{B_1^i} = \frac{1}{\sqrt{2}} \neq 0. \tag{16}$$

In bond dimension one, trace normalization fixes the positive diagonal factor to 1, while scalar conjugation cancels. Hence no square-root decomposition of these two BNT members can produce a joint residual-isometry family. This is the theorem `MPSTensor.cpsvCorollary312_arbitraryBNT_counterexample` in `TNLean/MPS/RFP/BNTResidualIsometryCounterexample`.

The full corrected conclusion holds for the active phase classes of literal CPSV canonical-form data. Write

$$A^i = U^\dagger \left( \bigoplus_k \mu_k A_k^i \right) U, \qquad UU^\dagger = I, \tag{17}$$

with every $A_k$ normal, and choose one representative $C_j$ from each phase class among the indices with $\mu_k \neq 0$. If $A$ has idempotent transfer map, then there are invertible matrices $X_j$, positive numbers $\lambda_{j,\alpha}$ with $\sum_\alpha \lambda_{j,\alpha} = 1$, and tensors $V_j$ such that

$$C_j^i = X_j \,\mathrm{diag}\Big(\sqrt{\lambda_{j,\alpha}}\Big)\, V_j^i X_j^{-1}, \tag{18}$$

$$\sum_i (V_j^i)_{\alpha,\beta} \overline{(V_{j'}^i)_{\alpha',\beta'}} = \delta_{j,j'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{19}$$

This statement includes the empty active family. It is formalized by `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.exists_residualIsometryFamily_of_isTransferIdempotent`. The predicate-level wrapper, which chooses the canonical witness data and its active refinement, is `MPSTensor.IsCPSVCanonicalForm.exists_activeBNT_residualIsometryFamily_of_isT` ransferIdempotent. Both declarations are in TNLean/MPS/RFP/CPSVCanonicalForm.

The proof first recovers the retained weighted direct sum through the ambient coisometry. For every active representative, MPSTensor.CPSVCanonicalFormData.active_weight_norm_one_and_block_rfp gives

$$|\mu_j| = 1, \qquad \mathcal{E}_{C_j}^2 = \mathcal{E}_{C_j}. \tag{20}$$

The normal-tensor characterization supplies the square-root decomposition. Next put each representative in a trace-preserving Perron gauge and keep its unit-modulus weight. Idempotence of the retained direct sum restricts to every mixed active pair. Distinct phase-class representatives are not gauge-phase equivalent, so the mixed spectral gap forces

$$\mathcal{E}_{\mu_j C_j, \mu_{j'} C_{j'}} = 0 \qquad (j \neq j'). \tag{21}$$

Cancelling the nonzero scalar factors and the invertible gauges gives $\mathcal{E}_{C_j, C_{j'}} = 0$. Removing the positive diagonal factors in the square-root decompositions and evaluating at matrix units yields the second display. Thus the simultaneous off-diagonal equations require no additional direct-sum hypothesis.

The source classification nevertheless remains *not-ready as printed.* The corrected theorem quantifies over representatives of the active phase classes, whereas Corollary 3.12 quantifies over an arbitrary BNT. The formal unused-member counterexample above refutes that printed quantifier.

# 7 Backward direction (multiplicity-one)

The backward direction of the characterization is also formalized at the normal-tensor-block level. If each block $B_k$ is in isometry canonical form and the cross-block mixed transfer operators vanish, $\mathcal{E}_{j,j'} = 0$ for $j \neq j'$, then the direct sum $\bigoplus_k B_k$ is a renormalization fixed point,[8] by MPSTensor.blockTransferSum_idempotent_of_isIsometryCanonicalForm. The direct-sum transfer map decomposes block by block (MPSTensor.blockDiagonal'_transferSum_toBlock): the diagonal blocks are the per-block transfer maps, idempotent by MPSTensor.isTransferIdempotent_of_isIsometryCanonicalForm, and the off-diagonal blocks vanish by the cross-block hypothesis, which is the $j \neq j'$ part of eq:III_isometry and so belongs to the source isometry form rather than being an added assumption.

This handles the distinct-blocks (multiplicity-one, phase-one) case of the source canonical form III_CFI_RFP, where the inner direct sum over copies $q = 1, \ldots, r_j$ has a single summand and the phases $\mu_{j,q}$ are all 1, so the direct sum carries one copy of each normal-tensor block. The full-multiplicity backward direction, summing repeated copies of each block with their phases and scalar weights, requires care about the physical direct sum carried by the copy index. A

[8] Formal declaration: MPSTensor.isTransferIdempotent_directSumTensor_of_isIsometryCanonicalForm in TNLean/MPS/RFP/ResidualIsometry.

literal virtual-block substitution need not satisfy the defining physical-isometry relation.

For a sector decomposition $P$ satisfying the single BNT canonical-form predicate, the explicit blockwise hypotheses are now eliminated. The theorem `MPSTensor.IsBNTCanonicalForm.isTransferIdempotent_basisDirectSum_iff` derives positive bond dimensions, normality, irreducibility, left-canonical normalization, and gauge-phase distinctness from that predicate and gives the preceding characterization for the direct sum of the basis representatives. Under the forward hypothesis, `MPSTensor.IsBNTCanonicalForm.exists_residualIsometryFamily_of_isTransferIdempotent_basisDirectSum` also supplies the trace-normalized decompositions and the full residual isometry family. This discharges the canonical-form extraction for the multiplicity-one basis direct sum. It does not identify that direct sum with the repeated-copy tensor carrying the weights $\mu_{j,q}$.

For a literal direct sum of any block family $(B_k)_k$, whole-tensor transfer idempotence has now been reduced exactly to the mixed block equations

$$\mathcal{E}^2_{\oplus_k B_k} = \mathcal{E}_{\oplus_k B_k} \quad \Longleftrightarrow \quad \mathcal{E}^2_{j,j'} = \mathcal{E}_{j,j'} \quad \text{for every } j, j'.$$

This is formalized by `MPSTensor.isTransferIdempotent_directSumTensor_iff_pairwise_mixedTransferMap_isIdempotentElem`. It shows that repeated virtual copies necessarily impose equations between distinct copies of the same normal tensor; checking only the representative block omits part of the whole-tensor fixed-point condition.

# 8 Ambiguity of a literal repeated-phase reading

Equation `III_CFI_RFP` allows a fixed normal tensor indexed by $j$ to occur several times, indexed by $q$, with independent coefficients $\mu_{j,q}$ satisfying $|\mu_{j,q}| = 1$. Take one physical letter, one scalar representative

$$B^0 = (1),$$

and two copies with $\mu_{1,1} = 1$ and $\mu_{1,2} = -1$. With trivial gauges and $\Lambda = (1)$, a literal reading of the displayed virtual block diagonal gives

$$A^0 = \operatorname{diag}(1, -1).$$

The representative $B$ satisfies the trace-normalized unit-isometry form: take $X = (1)$, $\Lambda = (1)$, and $U^0 = (1)$. Both copy phases have unit modulus. For $g = 1$ and a one-dimensional representative, the displayed joint equation `eq:III_isometry` also reduces to $|U^0|^2 = 1$ and is satisfied. Notice, however, that this equation carries the index $j$ but not the copy index $q$. These facts are formalized by `MPSTensor.scalarUnitTensor_isIsometryCanonicalForm` and `MPSTensor.phaseFlipTensor_is_two_phase_copies` in `TNLean/MPS/RFP/PhaseMultiplicityCounterexample`.

The prose following the display supplies additional meaning: the indices $j, q$ connect the physical and virtual indices and therefore give a direct sum in both spaces, up to the physical isometry (local source `Papers/1606.00608/MPDO-22-12-17-2.tex:559--563`). The one-letter tensor above realizes the virtual direct sum but cannot realize a nontrivial physical direct sum. Indeed,

$$(A^0)^2 = I,$$

and there is no scalar $v$ with $I = vA^0$: the two diagonal entries would require $v = 1$ and $v = -1$ simultaneously. Thus this tensor fails even the unrestricted scalar-coefficient version of the defining relation `AA=A`. This is formalized by `MPSTensor.phaseFlipTensor_no_blocking_coefficient`.

Its transfer map is

$$\mathcal{E}_A(Y) = A^0 Y (A^0)^\dagger.$$

For the matrix unit $E_{01}$ one has

$$\mathcal{E}_A(E_{01}) = -E_{01}, \qquad \mathcal{E}_A^2(E_{01}) = E_{01}.$$

Thus $\mathcal{E}_A^2 \neq \mathcal{E}_A$. The formal theorem `MPSTensor.phaseFlipTensor_not_isTransferIdempotent` proves that this tensor does not satisfy the present predicate `MPSTensor.IsTransferIdempotent`.

The calculation exposes a gap between the displayed theorem and its accompanying physical-space interpretation. The source definition of a pure-state RFP is the physical isometry relation `AA=A` (local source `Papers/1606.00608/MPDO-22-12-17-2.tex:420--425`). It is formalized by `MPSTensor.HasPhysicalBlockingIsometry`. The unrestricted equivalence between this relation and transfer-map idempotence is `MPSTensor.isTransferIdempotent_iff_hasPhysicalBlockingIsometry`; it is the Kraus-family argument at source lines 1205–1209. The source then states the repeated-copy form in Theorem `charact-MPS`, including the independent unit phases (local source `Papers/1606.00608/MPDO-22-12-17-2.tex:543--554`). For the two-phase virtual-block reading above, neither `AA=A` nor literal whole-tensor idempotence holds.

The displayed isometry equation at lines 551–554 contains $j$ but no $q$. The prose at lines 561–563 says that $j, q$ participate in physical and virtual direct sums, but it gives neither a $q$-indexed isometry nor an equation for such an isometry. In particular, a formal construction would also require a dimension condition for embedding the resulting direct sum into the physical space; no such condition occurs in the theorem. Adding these data to a Lean statement would therefore add hypotheses absent from the cited statement and would violate the faithfulness rule. The full repeated-copy theorem cannot be marked as formalized without a corrected source statement.

# 9 Elimination plan

The canonical-form extraction is discharged for the multiplicity-one basis direct sum by the two theorems cited above. For the literal repeated virtual block sum, the scaling law is

$$\mathcal{E}_{\mu B,\nu C} = \mu\overline{\nu}\,\mathcal{E}_{B,C}.$$

It is formalized by `MPSTensor.mixedTransferMap_smul`. Combining it with the pairwise mixed-transfer criterion proves that, for repeated copies of a nonzero transfer-idempotent block, whole-tensor idempotence forces $\mu_q\overline{\mu_{q'}} = 1$ and hence equality of all unit-modulus phases. This corrected literal-block theorem is `MPSTensor.phases_eq_of_isTransferIdempotent_directSum_scaled_self`. Thus the literal-block replacement is complete, but it is a correction rather than a formalization of the printed independent-phase theorem.

The independent-phase statement itself requires an author correction or an erratum specifying the missing $q$-dependent physical map, its isometry equation, and the necessary dimension hypothesis. No such data will be introduced as new assumptions merely to obtain a formal equivalence.

# 10 Conclusion

The per-block isometry form exposes two normalization conventions, which the formal development makes explicit. In the $D_k^{-1/2}$ convention the residual tensor satisfies the contracted per-block identity

$$\sum_i (U_k^i)^\dagger U_k^i = I$$

and its pair-index equation has right-hand side $D_k^{-1}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$. In the rescaled unit pair-index convention the residual tensor satisfies

$$\sum_i \overline{(U_k^i)_{\alpha,\beta}}\,(U_k^i)_{\alpha',\beta'} = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'},$$

while the contracted identity is instead $\sum_i (U_k^i)^\dagger U_k^i = D_k I$. Thus the scalar normalization comparison is explicit. For every normal tensor, whose bond dimension is necessarily positive, transfer idempotence is now equivalent to the trace-normalized square-root form by `MPSTensor.IsNormalTensor.isTransferIdempotent_iff_isIsometryCanonicalForm`. The diagonal weight is $\Lambda_\alpha = \rho_{\alpha,\alpha}/\operatorname{tr}\rho$, and $\sqrt{\Lambda}$ dresses the unit isometry. This closes the single-normal-tensor lemma under its printed hypotheses while preserving the local square-root correction. The cross-block $\delta_{j,j'}$ orthogonality equations of `eq:III_isometry` hold for a direct sum of distinct normal blocks whose whole transfer map is idempotent, by `MPSTensor.exists_residualIsometryFamily_of_isTransferIdempotent_directSum`. For literal CPSV canonical-form data, every active listed block has unit-modulus weight and is itself a renormalization fixed point. After choosing one representative from each active phase class, the trace-normalized square-root forms can be chosen simultaneously so that all diagonal and off-diagonal joint residual-isometry equations hold. This is `MPSTensor.CPSVCanonicalFormData.ActiveBNTRefinement.exists_residualIsometryFamily_of_isTransferIdempotent`, with predicate wrapper `MPST` `ensor.IsCPSVCanonicalForm.exists_activeBNT_residualIsometryFamily_of_isTransferIdempotent`. The printed arbitrary-BNT statement remains false: the bond-one pair $(B_0, B_1)$ above includes an unused BNT member and has cross inner product $1/\sqrt{2}$.

The remaining repeated-copy part of Theorem 3.11 is not presently a well-specified mathematical statement: its displayed isometry omits $q$, while the prose invokes an unspecified physical direct sum. The tensor $A^0 = \mathrm{diag}(1, -1)$ proves that the literal virtual-block reading fails both `AA=A` and whole-tensor transfer-map idempotence. The exact literal-block criterion is therefore recorded, while the independent-phase claim awaits a corrected source statement.

# References


[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.