

The Pure-State Blocking-up-to-Gauge Equivalence Is False

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Verdict: false-source (resolved).

1 The source claim

Appendix D of [CPGSV16], equation `RFP-gauge` and lines 2100–2107 of the local source, allows a renormalization step to introduce a gauge transformation on the virtual indices. The paragraph following the diagram claims that, for pure tensors in canonical form, this condition is equivalent to the ordinary renormalization fixed-point condition. Its first step asserts that similarity of the transfer maps $\mathcal{E}^2$ and $\mathcal{E}$ forces their eigenvalues to lie in $\{0, 1\}$.

That spectral inference is false. A finite spectrum can be invariant under squaring because its nonzero eigenvalues form a nontrivial root-of-unity cycle. Similarity of $\mathcal{E}^2$ and $\mathcal{E}$ does not imply $\mathcal{E}^2 = \mathcal{E}$.

Definition `RFPMixedTS`, lines 657–660, imposes canonical form on the MPDO tensor M . We use that literal reading of “in CF” in the formal refutation. Thus the canonical-form and weight-normalization hypotheses are proved for the doubled MPO tensor $M = A \otimes \bar{A}$, not merely for the underlying pure MPS tensor A .

2 A source-shaped counterexample

Let $\omega = e^{2\pi i/3}$ and consider the pure one-letter tensor

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}. \tag{1}$$

Let

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Since $\omega^3 = 1$, direct multiplication gives

$$A = \omega S A^2 S^{-1}. \tag{3}$$

This has the orientation of the printed diagram: the original tensor is the phase-scaled virtual conjugate of its two-site blocking. The corresponding pure MPDO tensor is the doubled tensor

$$M = A \otimes \overline{A}. \quad (4)$$

For a chain length $N \geq 1$, let $V^{(N)}(A)$ denote the MPS wavefunction generated by A . The tensor M generates the rank-one operator $|V^{(N)}(A)\rangle\langle V^{(N)}(A)|$, which is positive semidefinite. Thus M satisfies the MPDO standing hypothesis in Definition `RFPMixedTS`. To verify its canonical-form hypothesis, set $\mu_0 = 1$ and $\mu_1 = \omega$, and let I_1 denote the bond-one scalar unit tensor. After identifying a doubled bond coordinate with a pair $(p, q) \in \{0, 1\}^2$, the tensor decomposes as

$$M = \bigoplus_{p, q \in \{0, 1\}} \mu_p \overline{\mu_q} I_1. \quad (5)$$

Each copy of I_1 is normal because its transfer map is the identity, and $|\mu_p \overline{\mu_q}| = 1$. This gives CPSV canonical-form data whose four weights satisfy the source normalization convention.

For one physical letter, both physical matrix algebras are one-dimensional. Every unitary-conjugation channel on either algebra is therefore the identity, so the source's physical channels contribute no phase. Instead, the concrete MPS amplitude phase cancels when the tensor is doubled:

$$\begin{aligned} (\omega S A^{[2]} S^{-1}) \otimes (\overline{\omega} \overline{S} \overline{A}^{[2]} \overline{S}^{-1}) &= (\omega \overline{\omega}) (S \otimes \overline{S}) M^{[2]} (S \otimes \overline{S})^{-1} \\ &= (S \otimes \overline{S}) M^{[2]} (S \otimes \overline{S})^{-1}. \end{aligned} \quad (6)$$

Thus $M^{[2]}$ and M are related by the literal virtual gauge condition in both directions of the Appendix D diagram. The concrete MPS identity in (3) is used only to construct this exact doubled-MPDO witness.

For a bond matrix ρ , define the one-letter transfer map by

$$\mathcal{E}_A(\rho) = A \rho A^\dagger. \quad (7)$$

This transfer map is not idempotent. On the off-diagonal matrix unit e_{21} ,

$$\mathcal{E}_A(e_{21}) = \omega e_{21}. \quad (8)$$

A second application gives

$$\mathcal{E}_A^2(e_{21}) = \omega^2 e_{21}. \quad (9)$$

Because $\omega \neq 1$, these values differ. Equivalently, the dyadic powers $\mathcal{E}_A^{2^n}(e_{21})$ oscillate between the two nontrivial cube-root phases and cannot converge, whereas an idempotent transfer map has every positive power equal to itself.

The formal declarations are `MPOTensor.IsOneLetterRFPViaTSUpToVirtualGauge`, `MPSTensor.IsPureOneLetterRFPViaTSUpToVirtualGauge`, `MPSTen`

`sor.IsBlockedGaugePhaseFixedPoint`, `MPSTensor.doubledVirtualGauge`,
`MPSTensor.isPureOneLetterRFPViaTSUpToVirtualGauge_of_isBlockedGaugePhaseFixedPoint`, `MPSTensor.cube_phase_blocked_gauge_identity`,
`MPSTensor.cubePhaseDoubledTensor_isMPDO`, `MPSTensor.cubePhaseDoubledTensor_isCPSVCanonicalForm`, `MPSTensor.cubePhaseDoubledTensor_exists_weightNormalized`, `MPSTensor.cubePhaseTensor_isPureOneLetterRFPViaTSUpToVirtualGauge`, `MPSTensor.cubePhaseTensor_not_isTransferIdempotent`, and `MPSTensor.cpsv16_pure_rfp_gauge_equivalence_false` in `TNLean/MPS/RFP/GaugeBlockingCounterexample`.

3 Verdict

Verdict: false as printed. The pure-state equivalence following equation `RFP-gauge` fails even when the one-letter doubled MPO tensor generates an MPDO family, is canonical, and satisfies the line-246 normalization. The ordinary theorem that a virtual gauge preserves an already established renormalization fixed point is unaffected: gauge covariance cannot turn the blocking-up-to-gauge hypothesis into transfer-map idempotence.

The printed pure spectral argument therefore cannot support the mixed-state continuation in the next paragraph. This counterexample does not decide the mixed-state question.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.