Purification RFP in Cirac–Perez-Garcia–Schuch–Verstraete 2017

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Verdict: false-source (open).

1 Source statement

Cirac–Pérez-Garcia–Schuch–Verstraete define a purification renormalization fixed point for an MPDO tensor M through a purifying MPS tensor A:

$$\rho_p^{(N)}(M) = \mathrm{tr}_a\big(|\Psi^{(N)}(A)\rangle\langle\Psi^{(N)}(A)|\big).$$

The source definition says that M is a PRFP when it admits such a representation with A an RFP.[1]

The following theorem in the source states that, for tensors M satisfying the purification equation above, the three conditions

M is a PRFP, $\qquad M$ has ZCL, $\qquad \rho^{(N)}(M)$ has the displayed PRFP decomposition

are equivalent.[2] The later warning that this definition is too weak concerns saturation of the area law, not the implication from PRFP to zero correlation length.

2 Deleted formalization

The deleted predicate `MPOTensor.IsPRFP` did not formalize the source definition. It did require a local LPDO witness and the condition

$$\mathsf{MPSTensor.IsTransferIdempotent}(\mathsf{purifyingMPSTensor}(A)),$$

so the pure-state RFP requirement itself was present. The missing source data was the stronger purification-RFP structure around `eq:MPDO-Puri-1`: after tracing the ancillary index, the source obtains a trace-preserving completely

[1] Source location: `Papers/1606.00608/MPDO-22-12-17-2.tex`:750–763, Definition `def:Puri-RFP`.

[2] Source location: `Papers/1606.00608/MPDO-22-12-17-2.tex`:775–786.

positive map on the spin degrees of freedom and the decomposition displayed in `PuriRFP`–`PuriRFP-BNT`. The purifying tensor is also described by the canonical RFP form `III_CFI_RFP`, but this is not the condition violated by the deleted scalar witness; in bond dimension 1 the witness already has that form. The deleted predicate kept only the local purifying family together with generic transfer-map idempotence of the doubled MPS tensor. It did not encode the global purification equation or the trace-preserving post-ancilla map.

With this weaker hypothesis the development proved a counterexample to PRFP implying ZCL. That conclusion contradicted the source theorem above, so it was a counterexample to the local Lean predicate, not to Definition `def:Puri-RFP`.

For this reason the file `TNLean/MPS/MPDO/PRFP.lean`, the corresponding blueprint section, and the counterexample registry entry were removed. The coverage audit now records the source PRFP definition as open, not as formalized.[3]

3 Global formalization and its obstruction

A literal formalization of the printed purification RFP definition states the positive-length global purification equation `eq:MPDO-Puri-1` and require the purifying spin–ancilla tensor to be a pure-state RFP. The trace over the ancillary index, discussed after Definition `def:Puri-RFP`, is a separate one-site tpCPM; it is not the two-map mixed-state RFP condition `RFPMixedTS`. The definitions are now present in `TNLean/MPS/MPDO/PRFP.lean`: the global equation is `MPOTensor.HasGlobalPurificationEquation`, the pure-state witness is `MPOTensor.HasPurificationRFPWitness`, and the bare positive-length global predicate is `MPOTensor.IsPRFP`. The latter records the displayed equation and pure-state transfer idempotence. The ancillary-trace map is `MPOTensor.ancillaryTraceMap`; its Kraus-form trace-preserving complete positivity is `MPOTensor.ancillaryTraceMap_isKrausCPTP`, and the corresponding spin-reduction condition is `MPOTensor.HasTracePreservingSpinReduction`.

The positive-length global equation does not, however, determine the one-site MPO tensor. This prevents a global-to-local theorem even after imposing MPDO positivity and a nonzero physical-trace transfer. Let $d = 1$, $D = 3$, and let the only tensor entry be

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For every $N > 0$, one has $\mathrm{tr}(Q^N) = 1$. Thus the generated density operator is the scalar 1, which is the purification density of the scalar pure-state RFP tensor $A = 1$. Every generated MPO is positive semidefinite and the physical-trace

[3] This note records the divergence removed in PR #2386. The faithful replacement is tracked by issue #2383 and the MPDO coverage tracker issue #2380.

transfer is $Q \neq 0$. Nevertheless

$$Q^2 = \mathrm{diag}(1, 0, 0)$$

is not a positive scalar multiple of Q, since its $(2, 3)$ entry vanishes while the corresponding entry of Q is 1. Hence source ZCL fails. The theorem `MPOTensor.exists_isPRFP_isMPDO_physTraceTransfer_ne_zero_not_isSourceZCL` formalizes this example.

Consequently nondegeneracy cannot repair the printed global statement. The source may instead intend the local purification graphic `Psipuri`, or a minimal or canonical representative excluding trace-invisible nilpotent bond sectors; neither condition is stated in Definition `def:Puri-RFP`. No such additional condition is introduced here. The global-to-local implication and the PRFP–ZCL equivalence therefore require clarification of the source statement rather than a proof from its present hypotheses.

4 Status

A separate local predicate `MPOTensor.IsLocalPurificationRFP` is now in `TNLean/MPS/MPDO/LocalPurificationRFP.lean`. It records the tensor-level local purification structure

$$M^{ij} = \left(\sum_k A^{(i,k)} \otimes \overline{A^{(j,k)}} \right)_e$$

together with the requirement that the purifying tensor A, viewed as an MPS tensor on the combined spin–ancilla index, is a pure-state renormalization fixed point (`MPSTensor.IsTransferIdempotent`). This predicate pins the local tensor of M and proves the useful consequence

$$\mathsf{MPOTensor.IsLocalPurificationRFP.isMPDO} : \mathsf{MPOTensor.IsLocalPurificationRFP}(M) \Longrightarrow \mathsf{MPOTensor.IsMPDO}(M$$

via `MPOTensor.IsLPDO.isMPDO`.

This local predicate is not Definition `def:Puri-RFP`. The source-level positive-length global purification equation and the trace-preserving structure appearing after the ancillary trace are recorded separately in `TNLean/MPS/MPDO/PRFP.lean`. The nondegenerate predicate `MPOTensor.IsNondegeneratePRFP` combines the local purification-RFP identity with $\mathcal{T}_M \neq 0$. The theorem `MPOTensor.isNondegeneratePRFP_iff` proves

$$\mathrm{NondegeneratePRFP}(M) \Longleftrightarrow \big(\mathrm{LPDO}(M), \quad \mathcal{T}_M \neq 0, \quad \mathcal{T}_M^2 = \mathcal{T}_M\big)\,.$$

This is an exact structural characterization of the nondegenerate local purification predicate. It is not the source PRFP–ZCL equivalence of Theorem 4.4. The forward implication from this stronger local predicate to source zero correlation length is proved below. The converse from source zero correlation length remains open. The passage from a tensor known only through the positive-length global equation to the local tensor identity is false by the nilpotent example above.

5 Normalized local equivalence on the physical-trace transfer

The forward implication of the source theorem (lines 775–786), from a purification renormalization fixed point to zero correlation length, *restricted* to the local purification predicate `MPOTensor.IsLocalPurificationRFP` (the one-site tensor identity $M^{ij} = (\sum_k A^{(i,k)} \otimes \overline{A^{(j,k)}})_e$, not the global Definition `def:Puri-RFP` at line 758) and to the physical-trace-transfer notion of zero correlation length, is formalized as below; the unrestricted global implication is false for the printed hypotheses (see the preceding section). That the object $\mathcal{T}_M = \sum_i M^{ii}$ (Definition 4.2, lines 735–739) is the source zero-correlation-length object is the verdict of the companion note `cpsv16_zcl_canonical_form_normalization.tex`. The theorem `MPOTensor.physTraceTransfer_sq_of_isLocalPurificationRFP` proves that the local purification renormalization fixed point condition makes $\mathcal{T}_M$ idempotent, $\mathcal{T}_M \mathcal{T}_M = \mathcal{T}_M$, with no extra hypothesis: the transfer matrix of the pure-state purifying tensor is idempotent, and closing the physical legs of M transports this idempotence through the bond identification and the Kronecker factor swap.

The converse transport is also valid. The theorem `MPOTensor.purificationTensor_isTransferIdempotent_iff_physTraceTransfer_sq` proves, for a fixed local purification,

$$\mathsf{MPSTensor.IsTransferIdempotent}(\widehat{A}) \iff \mathcal{T}_M^2 = \mathcal{T}_M.$$

Both the bond identification and the Kronecker-factor swap are bijective reindexings, so idempotence transports in both directions; faithfulness of the transfer-matrix representation recovers transfer-map idempotence. Existentially packaging the purifying data gives `MPOTensor.isLocalPurificationRFP_iff_isLPDO_and_physTraceTransfer_sq`.

The corresponding local conclusion that M has source zero correlation length, `MPOTensor.isSourceZCL_of_isLocalPurificationRFP`, carries the extra hypothesis $\mathcal{T}_M \neq 0$. This records a normalization restriction: the unconditional implication is false. The zero tensor at $D = D' = d_K = 1$ with $A^{(i,0)} = 0$ satisfies the local purification condition, since its purifying tensor is identically zero and $0 \circ 0 = 0$ makes that tensor a pure-state renormalization fixed point, yet $\mathcal{T}_M = 0$, so the conclusion fails. The hypothesis $\mathcal{T}_M \neq 0$ reflects the source purification renormalization fixed point being a normalized pure-state fixed point with leading eigenvalue 1, a normalization that `MPOTensor.IsLocalPurificationRFP` drops. The theorem `MPOTensor.exists_isPRFP_not_isSourceZCL` now records this zero-purifier obstruction for the bare global predicate itself. The stronger theorem `MPOTensor.exists_isPRFP_isMPDO_physTraceTransfer_ne_zero_not_isSourceZCL` shows that global MPDO positivity and $\mathcal{T}_M \neq 0$ still do not suffice: positive-length traces do not see the nilpotent bond sector of Q.

6 Remaining density-form obstruction

The third clause of the source theorem, equations `PuriRFP`– `PuriRFP-BNT` at lines 765–773, expands the purifying pure-state RFP into its canonical repeated-copy form. A full formalization of this clause cannot currently be obtained from the printed statement without adding data. The underlying pure-state display `III_CFI_RFP` permits independent phases $\mu_{j,q}$, while the physical isometry equation `eq:III_isometry` has no copy index q and supplies no dimension condition for a copy-dependent physical direct sum. The literal virtual-block reading is false, as proved by the two-phase counterexample recorded in `docs/paper-gaps/cpsv16_rfp_isometry_scope.tex` and closed issue #2598.

Consequently the normalized local PRFP-to-ZCL implication is complete. The converse implication required for the source equivalence remains open, while the global-to-local passage from Definition 4.3 is disproved under the printed hypotheses. The repeated-copy density formula remains source-obstructed pending an author correction or erratum. No copy-indexed physical map, minimality condition, or additional structural hypothesis should be introduced merely to force either printed claim.

References