A raw-weight counterexample to the pure-state ZCL equivalence in Cirac–Perez-Garcia–Schuch–Verstraete 2017

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Verdict: false-source (resolved).

1 Source statement and failed inference

The canonical form of [CPGSV16] permits repeated copies of a normal tensor with coefficients of modulus at most one, provided that at least one coefficient equals one. In the notation of equation `eq:II_ABasicTensors`,

$$A^i = \bigoplus_{j=1}^{g} \bigoplus_{q=1}^{r_j} \mu_{j,q} X_{j,q} A_j^i X_{j,q}^{-1}.$$

Theorem 3.8, labelled `TheoremZCLPure` in the local source, asserts that a canonical-form tensor has zero correlation length if and only if its transfer matrix is idempotent.

The converse argument begins at source lines 1248–1251. It infers from $\mathbb{E}_A^2 \neq \mathbb{E}_A$ that some normal basis tensor A_j has a non-idempotent transfer matrix $\mathbb{E}_{j,j}$. This inference omits the copy coefficients. On the $(j, q; j', q')$ bond block, the full transfer matrix is

$$\mu_{j,q}\overline{\mu_{j',q'}}\,\mathbb{E}_{j,j'}.$$

Thus the full transfer matrix can have an eigenvalue strictly between zero and one even when every $\mathbb{E}_{j,j}$ is idempotent.

2 Counterexample

Take one physical letter and the scalar normal tensor $C^0 = (1)$. Form the canonical tensor. In the sector-adapted coordinates of this decomposition,

$$A^0 = C^0 \oplus \tfrac{1}{2} C^0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

This has one BNT component, two raw copies, and weights 1 and 1/2. The weights satisfy precisely the source normalization: both have modulus at most one and the first equals one. The matrix-product-vector coefficient at length N is

$$V^{(N)}(A) = 1 + 2^{-N},$$

which is also the coefficient obtained from the two-copy BNT decomposition.

There is only one BNT component, so local orthogonality is vacuous. The physical Hilbert space of every finite region is one-dimensional. Every physical observable is therefore scalar, and every two-region expectation is independent of the separation. Hence A satisfies the CID and local orthogonality conditions of Definition 3.6.

The transfer map has the matrix units as eigenvectors, with eigenvalues

$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{2}, \quad \tfrac{1}{4}.$$

It is not idempotent. Equivalently, the one-letter blocking equation required at a renormalization fixed point would give a scalar v such that

$$(A^0)^2 = vA^0.$$

The $(0,0)$ entry gives $v = 1$, while the $(1,1)$ entry would then assert $1/4 = 1/2$.

All four assertions have kernel-checked proofs.[1]

3 Consequence for the theorem surface

The unrestricted equivalence in Theorem 3.8 is false under the stated raw-weight canonical-form normalization. The example disproves the zero-correlation-length implication to transfer idempotence. It does not settle the converse implication for adjacent physical regions; the existing formal result in that direction assumes both complementary gaps are positive.

A corrected theorem must remove the copy-weight eigenvalues from its conclusion. Two possible formulations are to pass first to a canonical representative with one unit-weight copy of each BNT component, or to state an asymptotic assertion for the limiting transfer projector rather than literal idempotence of the raw tensor's transfer map. A restriction merely requiring unit-modulus copy weights is insufficient: distinct phases can also destroy idempotence. Any replacement must therefore state explicitly how repeated copies and their phases are treated.

The first formulation is now proved. For the direct sum containing one unit-weight copy of each distinct BNT component, positive-gap physical CID together with BNT local orthogonality is equivalent to transfer idempotence. The reverse implication uses arbitrary sector-supported rank-one observables and nondegeneracy of the trace pairing, thereby avoiding the false nonzero-subleading-eigenvalue inference at source line 1250. This result does not apply to

[1] Formal conjunction: `MPSTensor.halvedWeightTensor_counterexample_to_unrestricted_zcl_iff_rfp` in `TNLean/MPS/RFP/BNTWeightCounterexample`.

the raw weighted tensor above and therefore does not change the counterexample verdict.

Issue GitHub issue #2597 should not retain the unrestricted biconditional as a proof target. Its replacement should distinguish the false raw-tensor statement from any theorem about a selected representative of the generated matrix-product-vector family.

The verdict is a source counterexample: the zero-correlation-length implication to transfer idempotence is false under the normalization stated in the cited canonical form. It is not an open formalization gap.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.