Pure zero correlation length and local orthogonality in Cirac–Perez-Garcia–Schuch–Verstraete 2017

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Verdict: scope-restriction (open).

1 Source statement

In Section 3.2 of [CPGSV16], correlations independent of distance are defined by requiring, for separated observables, that translating one observable through any allowed separation does not change the expectation value.[1] The same subsection defines local orthogonality for a basis-of-normal-tensors family by the mixed-sector equations

$$\mathbb{E}_{j,j'} = \sum_i A^i_j \otimes \overline{A^i_{j'}} = 0, \qquad j \neq j', \tag{LO}$$

and then defines zero correlation length as the conjunction of CID and local orthogonality.[2]

For a single tensor A, write

$$\mathbb{E}_A = \sum_i A^i \otimes \overline{A^i} \tag{1}$$

for the transfer matrix.

The pure-state theorem states that, for a tensor A in canonical form,

$$\mathrm{ZCL}(A) \quad \Longleftrightarrow \quad \mathbb{E}_A^2 = \mathbb{E}_A. \tag{2}$$

In the proof, the source first identifies local orthogonality with the equations $\mathbb{E}_{j,j'} = 0$ for $j \neq j'$. After assuming those equations, it proves that $\mathbb{E}_A^2 = \mathbb{E}_A$ is equivalent to CID.[3]

[1]Local source: `Papers/1606.00608/MPDO-22-12-17-2.tex:437--445`, label `ZCL`.
[2]Local source: `Papers/1606.00608/MPDO-22-12-17-2.tex:467--479`.
[3]Local source: `Papers/1606.00608/MPDO-22-12-17-2.tex:1248--1268`.

2 Formal boundary

The current formal predicate `MPSTensor.IsLocallyOrthogonal` is a one-tensor convention:

$$\mathsf{MPSTensor.IsLocallyOrthogonal}(A) \iff \mathbb{E}_A^2 = \mathbb{E}_A. \tag{3}$$

Consequently

$$\mathsf{MPSTensor.IsZCL}(A) \iff \big(\mathbb{E}_A^2 = \mathbb{E}_A\big) \wedge \mathrm{CID}(A). \tag{4}$$

The theorem `MPSTensor.zcl_iff_idempotent_transfer` therefore proves the single-block equivalence

$$\big(\mathbb{E}_A^2 = \mathbb{E}_A\big) \wedge \mathrm{CID}(A) \iff \mathbb{E}_A^2 = \mathbb{E}_A. \tag{5}$$

Its nontrivial direction is the implication

$$\mathbb{E}_A^2 = \mathbb{E}_A \Longrightarrow \mathrm{CID}(A),$$

obtained from $\mathbb{E}_A^n = \mathbb{E}_A$ for all $n \geq 1$. The reverse direction of that displayed equivalence is immediate because the left-hand side already contains $\mathbb{E}_A^2 = \mathbb{E}_A$.

This is a useful local statement, but it is not the source theorem at lines 498–502 and 1248–1268. The source theorem uses unrestricted physical CID together with the BNT-family mixed-sector equations (LO). The source predicates now have formal counterparts. The predicate `MPSTensor.IsPhysicalCID` quantifies over two nonempty physical blocks and arbitrary natural complementary gap lengths, including zero, and requires the two-region expectation to be unchanged when the gaps are redistributed with fixed sum. The predicate `MPSTensor.IsPhysicalBNTZCL` is the conjunction of the BNT relation, this unrestricted physical CID condition, and BNT local orthogonality.

These new predicates state Definitions 3.3 and 3.6; they do not prove the equivalence in Theorem 3.8. In particular, the theorem `MPSTensor.zcl_iff_idempotent_transfer` still concerns the older single-block and virtual-insertion predicates.

3 Elimination plan

The formalization has a BNT-family version of local orthogonality, `MPSTensor.IsBNTLocallyOrthogonal`, carrying the equations $\mathbb{E}_{j,j'} = 0$ for all $j \neq j'$. Writing $\mathrm{BNT}(A)$ for the condition that the family is a basis of normal tensors for A, the source predicate is now

$$\mathrm{ZCL}_{\mathrm{BNT}}(A) := \mathrm{BNT}(A) \wedge \mathrm{CID}_{\mathrm{physical}}(A) \wedge \big(\forall j \neq j',\, \mathbb{E}_{j,j'} = 0\big).$$

Its formal name is `MPSTensor.IsPhysicalBNTZCL`. The older predicate `MPSTensor.IsBNTZCL` uses the virtual-insertion condition,

$$\mathrm{ZCL}_{\mathrm{BNT}}^{\mathrm{virt}}(A) := \mathrm{BNT}(A) \wedge \mathrm{CID}_{\mathrm{virt}}(A) \wedge \big(\forall j \neq j',\, \mathbb{E}_{j,j'} = 0\big),$$

where $\mathrm{CID}_{\mathrm{virt}}$ quantifies over arbitrary virtual bond matrices. The positive-gap formal predicate uses the physical condition below, [4]

$$\mathrm{ZCL}^{\mathrm{pos}}_{\mathrm{BNT}}(A) := \mathrm{BNT}(A) \wedge \mathrm{CID}^{\mathrm{pos}}_{\mathrm{physical}}(A) \wedge \big(\forall j \neq j',\, \mathbb{E}_{j,j'} = 0\big),$$

where $\mathrm{CID}^{\mathrm{pos}}_{\mathrm{physical}}$ is the positive-gap physical predicate.[5] The unrestricted predicate implies this positive-gap restriction by `MPSTensor.isPositiveGapPhysicalCID_of_isPhysicalCID`; the analogous BNT implication is `MPSTensor.isPositiveGapBNTZCL_of_isPhysicalBNTZCL`.

The unrestricted BNT-level equivalence is not a remaining proof target. Under the source's raw-weight canonical-form normalization it is false. The tensor

$$A^0 = \mathrm{diag}(1, 1/2)$$

has one normal BNT component with raw weights 1 and 1/2. It satisfies physical CID, while local orthogonality is vacuous, but its transfer matrix is not idempotent. The counterexample and the failed inference at source lines 1248–1251 are recorded in `docs/paper-gaps/cpsv16_pure_zcl_raw_weight_counterexample.tex`.[6]

The existing single-block theorem remains a scope restriction: it records the one-block idempotence/CID consequence and does not assert the false raw-weight BNT equivalence.

There is a second distinction in the correlation predicate. Definition 3.3 quantifies over physical observables on arbitrary nonempty finite regions, and the two-observable formula at lines 490–496 inserts such an observable through

$$\mathbb{E}_O(X) = \sum_{\boldsymbol{i},\boldsymbol{j}} \langle \boldsymbol{j}|O|\boldsymbol{i}\rangle A^{\boldsymbol{i}} X (A^{\boldsymbol{j}})^\dagger.$$

For two observed blocks on a periodic chain, the contraction is the operator trace on the virtual matrix space,

$$\mathrm{Tr}_{M_D(\mathbb{C})}\big(\mathbb{E}_{O_2} \circ \mathbb{E}_A^{n_2} \circ \mathbb{E}_{O_1} \circ \mathbb{E}_A^{n_1}\big)\,.$$

It is not the matrix trace obtained by applying this composition to the identity matrix; the latter gives a different contraction for a general Kraus family.

The formal positive-gap predicate records this formula when both complementary gaps between the observed regions are positive. The corresponding theorem proves[7]

$$\mathbb{E}_A^2 = \mathbb{E}_A \quad \Longrightarrow \quad \mathrm{CID}^{\text{positive gaps}}_{\mathrm{physical}}(A).$$

The positive-gap predicate is not Definition 3.3, which quantifies over all disjoint regions and therefore includes adjacent regions. The unrestricted predicate

[4] The formal predicate is `MPSTensor.IsPositiveGapBNTZCL`.

[5] The formal predicate is `MPSTensor.IsPositiveGapPhysicalCID`.

[6] Tracked by GitHub issue #2597.

[7] The formal theorem is `MPSTensor.isPositiveGapPhysicalCID_of_isTransferIdempotent`.

`MPSTensor.IsPhysicalCID` now records Definition 3.3. For an adjacent pair, one complementary transfer power in the formula at lines 490–496 is $\mathbb{E}_A^0 = 1$. Idempotence gives $\mathbb{E}_A^n = \mathbb{E}_A$ only for $n \geq 1$ and does not identify $\mathbb{E}_A^0$ with $\mathbb{E}_A$. The positive-gap theorem is consequently a scope restriction, not the full forward implication of the source theorem at lines 498–502. The older predicate `MPSTensor.IsCID` instead quantifies directly over arbitrary virtual matrices X, Y. Its converse to idempotence is valid, but it is stronger than physical CID. For the multiplicity-one unit-weight direct sum, the simultaneous block-injectivity theorem now supplies the corresponding physical observables from source lines 1250–1258. More precisely, `MPSTensor.IsBNTCanonicalForm.exists_basis_physicalObservableTransfer_rankOne_le_three_totalDim_pow_five` selects any one BNT basis sector and realizes the insertion $|R)(l|$ there on a region of size at most $3D^5$, with zero insertion on every other sector pair. This theorem acts on `directSumTensorP.basis`, not the raw weighted repeated-copy tensor `P.toTensor`. The source equations contain all copy pairs and their weight powers, so their unrestricted realization remains separate.

A further obstruction occurs in the preceding sentence at source line 1250. The source claims that $\mathbb{E}_1^2 \neq \mathbb{E}_1$ implies a nonzero eigenvalue λ with $|\lambda| < 1$. This implication is false for a non-idempotent operator whose only subleading spectral value is zero but whose generalized zero-eigenspace has a nontrivial nilpotent Jordan part. Consequently the unconditional reverse direction is not obtained from the printed proof, even for the multiplicity-one unit-weight representative. The conditional theorems with suffix `of_spectral_pair` therefore retain exactly the nonzero subleading left and right eigenvectors used in lines 1250–1262 and remain the formal record of the cited proof.

Local fix at line 1250. For the multiplicity-one, unit-weight representative, the false spectral inference can be bypassed without an additional hypothesis. Fix a BNT sector j, let ρ_j be its trace-one stationary state, and choose arbitrary $u, v \in M_{D_j}(\mathbb{C})$. The simultaneous physical-observable realization gives two sector-supported rank-one insertions

$$\mathbb{E}_{O_1} = |\rho_j)(u|, \qquad \mathbb{E}_{O_2} = |v)(1|.$$

Sector compression, stationarity of ρ_j, and the operator trace of a rank-one map give, for all complementary gap lengths n_1, n_2,

$$\mathrm{Tr}_{M_D(\mathbb{C})}\left(\mathbb{E}_{O_2}\mathbb{E}_A^{n_2}\mathbb{E}_{O_1}\mathbb{E}_A^{n_1}\right) = \mathrm{tr}\left(u\mathbb{E}_j^{n_1}(v)\right).$$

There is no transpose in this identity, and the right-hand side is independent of n_2. Positive-gap CID at $(n_1, n_2) = (1, 2)$ and $(2, 1)$ therefore gives

$$\mathrm{tr}\left(u\mathbb{E}_j(v)\right) = \mathrm{tr}\left(u\mathbb{E}_j^2(v)\right)$$

for every u, v. Nondegeneracy of the trace pairing yields $\mathbb{E}_j^2 = \mathbb{E}_j$. Local orthogonality eliminates the off-diagonal mixed transfer maps, so the multiplicity-one direct sum is idempotent.

This repair requires no diagonalizability, semisimplicity at zero, or nonzero subleading eigenvalue. It is formalized by `MPSTensor.IsBNTCanonicalForm.isTransferIdempotent_basisDirectSum_of_isPositiveGapBNTZCL`. It is a local correction to line 1250 rather than a formalization of the printed spectral argument. It does not alter the raw-weight counterexample or the adjacent-gap scope restriction.

Combining this repair with the fixed-point characterization of the all-chain commuting parent ground spaces gives the corrected representative-level equivalence, formalized by `MPSTensor.IsBNTCanonicalForm.isPositiveGapBNTZCL_basisDirectSum_iff_hasNNCPHGroundSpaces`. The statement concerns the multiplicity-one, unit-weight basis direct sum and positive complementary gaps. Repeated copies, arbitrary raw weights, and zero complementary gaps remain excluded.

Since the older predicate universally quantifies over positive definite fixed points, it is also vacuous for a tensor without such a fixed point. The periodic-chain predicate avoids this boundary-state issue. For the unrestricted raw-weight statement there is an additional obstruction: the step from non-idempotence of the full transfer matrix to non-idempotence of a normal BNT component ignores the eigenvalues contributed by the raw copy weights.

Accordingly, `MPSTensor.IsBNTZCL` remains the virtual-insertion surrogate, while `MPSTensor.IsPositiveGapBNTZCL` names the physical positive-gap condition. Neither is Definition 3.6; that definition is now `MPSTensor.IsPhysicalBNTZCL`. Its unrestricted equivalence with transfer idempotence is disproved by the raw-weight counterexample above.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.