

Length-Dependent Scalars Do Not Determine Copy Weights in the Proportional BNT Theorem

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Verdict: clarification (resolved).

1 The source statement

Cirac–Pérez-García–Schuch–Verstraete state in [CPGSV16, Theorem 2.10 and Appendix A, lines 1167–1183] that proportional matrix-product-vector families in canonical form have bijectively matched bases of normal tensors. For each matched pair there are a phase ζ_k and an invertible matrix X_k such that

$$Q_k^i = \zeta_k X_k P_{\beta(k)}^i X_k^{-1}. \quad (1)$$

The theorem does not assert that the two total tensors are conjugate, even up to a common phase. Copy-weight recovery and the block-diagonal global gauge occur in the equal-MPV Corollary 2.11, not in the proportional theorem.

The formal proportional theorem uses eventual nonzero projective proportionality: for all sufficiently large N , one nonzero scalar c_N satisfies

$$V^{(N)}(P_{\text{tot}}) = c_N V^{(N)}(Q_{\text{tot}}). \quad (2)$$

The source assumption at every positive length is a special case.¹

2 The coefficient identity retaining the global scalar

Write

$$c_N^{(j)}(P) = \sum_q \mu_{j,q}^N, \quad c_N^{(k)}(Q) = \sum_q \nu_{k,q}^N. \quad (3)$$

¹The formal proportional theorem is `MPSTensor.fundamentalTheorem_proportional_canonicalForm` in [TNLean/MPS/FundamentalTheorem/SectorBNT/FundamentalCoord](#).

Substituting the matched-sector relation (1) into (2), reindexing by β , and applying eventual linear independence of the P -basis gives

$$c_N^{(\beta(k))}(P) = c_N \zeta_k^N c_N^{(k)}(Q) \quad (4)$$

for every sector k and all sufficiently large N .² The same c_N occurs in every sector because it is the scalar relating the two total vectors.

The equal-MPV argument has $c_N = 1$. It then compares the two finite power sums

$$\sum_q \mu_{\beta(k),q}^N = \sum_q (\zeta_k \nu_{k,q})^N \quad (5)$$

and recovers the copy-weight multiset. For a general proportional family, (4) does not supply this scalar-free identity.

3 Counterexample

Let C be a normalized normal tensor and consider the one-sector BNT forms

$$P = C \oplus \frac{1}{2}C, \quad Q = C \oplus \frac{1}{3}C. \quad (6)$$

Both forms obey the canonical weight normalization: every weight has modulus at most 1, and each form has a unit-modulus weight. Their MPV families are

$$V^{(N)}(P) = (1 + 2^{-N})V^{(N)}(C), \quad V^{(N)}(Q) = (1 + 3^{-N})V^{(N)}(C). \quad (7)$$

Thus they are nonzero and proportional at every positive length with

$$c_N = \frac{1 + 2^{-N}}{1 + 3^{-N}}. \quad (8)$$

This sequence is not of the form ξ^N for a fixed phase ξ .

For a one-dimensional choice of C , the two total matrices have eigenvalue multisets

$$\{1, \frac{1}{2}\} \quad \text{and} \quad \{1, \frac{1}{3}\}. \quad (9)$$

Similarity preserves eigenvalues, and multiplication by a common phase merely multiplies both eigenvalues by that phase. Consequently the two total tensors are not conjugate, even up to a common phase.

²The formal coefficient identity that retains the global scalar is `MPSTensor.coeff_identity_via_matched_mpv_phase_proportional` in [TNLean/MPS/FundamentalTheorem/SectorBNT/CoeffIdentity](#).

4 Conclusion

Verdict: correction of a prior mathematical assessment; there is no mathematical gap. The source proportional theorem never requires copy-weight recovery or a global gauge, and the counterexample proves that this stronger claim is false under the proportional theorem’s hypotheses. In particular, there is no missing theorem asserting that the proportionality scalar of two BNT canonical forms must be a geometric phase sequence. The valid conclusions are:

- proportional total MPVs determine a bijection and gauge-phase equivalence of the normal sectors;
- the coefficient identity retains one arbitrary length-dependent global scalar c_N ;
- scalar-free coefficient identities, copy-weight matching, and a global block gauge require additional hypotheses;
- in the equal-MPV corollary, $c_N = 1$, so power-sum recovery and the global gauge follow.

The conditional implications from scalar-free coefficient identities to a global gauge remain correct.³ They are additional consequences, not unfinished parts of CPSV16 Theorem 2.10.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.

³The formal conditional global-gauge construction is `MPSTensor.ft_sector_bnt_proportional_global_gauge_of_coeff_identity` in [TNLean/MPS/FundamentalTheorem/SectorBNT/Fundamental](#).