

The SAL–ZCL Blocking-Channel Conclusion of CPSV16 Proposition C.15 Is False

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Verdict: false-source (resolved).

1 Source claim

In Appendix C.2 of Ref. [CPGSV16], a tensor K is fixed which generates matrix product density operators and is both simple and in block-injective canonical form. Its basis of normal tensors is written $(\mathcal{K}_j)_j$. Zero correlation length first makes the repeated-copy weights independent of the copy label, after which their common values are absorbed into the representatives $\mathcal{K}_j$. These are the standing hypotheses at source lines 1628–1665.

For an MPO tensor A of virtual bond dimension D and a virtual boundary $X \in M_D(\mathbb{C})$, write $\mathcal{K}_N^A(X)$ for the N -site physical matrix obtained by multiplying the local matrices around the chain and closing the virtual indices against X . Proposition C.15 claims that saturation of the area law and zero correlation length imply the existence of two trace-preserving completely positive maps satisfying the diagram C.24. In particular, one of these maps must obey

$$\mathcal{T}(\mathcal{K}_2^K(X)) = \mathcal{K}_3^K(X) \quad \text{for every } X \in M_D(\mathbb{C}). \quad (1)$$

The closing proof of Theorem 4.9 applies this map twice. Its final sentence labels this step $(iii) \Rightarrow (v)$, whereas the theorem’s displayed chain and the hypothesis of Proposition C.15 place it in the logical $(ii) \Rightarrow (v)$ route. This separate cyclic-label discrepancy is recorded in `docs/paper-gaps/cpgsv17_mpdo_theorem_4_9_implication_label.tex`. In either reading, the argument requires a trace-preserving completely positive map satisfying

$$\mathcal{T}^{(2)}(\mathcal{K}_2^K(X)) = \mathcal{K}_4^K(X) \quad \text{for every } X \in M_D(\mathbb{C}). \quad (2)$$

The source asserts these conclusions at lines 1810–1825 of Ref. [CPGSV16].

2 A four-letter trace-norm obstruction

Let $l_i \in \mathbb{C}^2$ and $r_i \in (\mathbb{C}^2)^*$, for $0 \leq i < 4$, be the columns and rows of

$$L := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -7 & 4 \end{pmatrix}, \quad (3)$$

$$R := \begin{pmatrix} 1/4 & -3/100 \\ 1/4 & -1/100 \\ 1/4 & 1/100 \\ 1/4 & 3/100 \end{pmatrix}. \quad (4)$$

Define $A_i = |l_i\rangle\langle r_i|$ and let A be the four-letter diagonal MPO tensor whose only nonzero physical slices are the matrices A_i . The two rectangular products satisfy

$$LR = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (5)$$

$$T := RL = \begin{pmatrix} 11/50 & 19/100 & 23/50 & 13/100 \\ 6/25 & 23/100 & 8/25 & 21/100 \\ 13/50 & 27/100 & 9/50 & 29/100 \\ 7/25 & 31/100 & 1/25 & 37/100 \end{pmatrix}. \quad (6)$$

Every entry of T is strictly positive, and direct multiplication gives

$$T^2 = \frac{1}{4}J_4, \quad (7)$$

$$T^3 = T^2, \quad (8)$$

where J_4 is the four-by-four all-ones matrix.

Choose the virtual boundary

$$X := \begin{pmatrix} -1 & 2/5 \\ -5 & 0 \end{pmatrix}. \quad (9)$$

Its endpoint pairing matrix is

$$D := (RXL)^\top = \begin{pmatrix} 0 & -1/10 & -1/5 & -3/10 \\ 1/10 & 0 & -1/10 & -1/5 \\ -4/5 & -9/10 & -1 & -11/10 \\ 3/10 & 1/5 & 1/10 & 0 \end{pmatrix}. \quad (10)$$

Since $A_i A_j = T_{ij} |l_i\rangle\langle r_j|$, every closure $\mathcal{K}_N^A(X)$ is diagonal. Its diagonal entry at $(i_1, \dots, i_N)$ is

$$(\mathcal{K}_N^A(X))_{(i_1, \dots, i_N), (i_1, \dots, i_N)} = D_{i_1, i_N} \prod_{q=1}^{N-1} T_{i_q, i_{q+1}}. \quad (11)$$

Consequently its trace norm is the sum of the absolute values of these real diagonal entries. Using Eqs. 7 and 8, the three relevant values are

$$\|\mathcal{K}_2^A(X)\|_1 = \frac{337}{250}, \quad (12)$$

$$\|\mathcal{K}_3^A(X)\|_1 = \frac{27}{20}, \quad (13)$$

$$\|\mathcal{K}_4^A(X)\|_1 = \frac{27}{20}. \quad (14)$$

A trace-preserving positive map contracts the trace norm on Hermitian matrices. The matrix $\mathcal{K}_2^A(X)$ is Hermitian. Thus any channel satisfying either the two-to-three or two-to-four closure identity would give

$$\frac{27}{20} \leq \frac{337}{250}, \quad (15)$$

which is impossible because

$$\frac{27}{20} - \frac{337}{250} = \frac{1}{500} > 0. \quad (16)$$

3 The ambient simple tensor

It remains to place this obstruction inside the precise standing hypotheses of Proposition C.15. Let $V : \mathbb{C}^4 \rightarrow \mathbb{C}^5$ be the coordinate isometry onto the first four physical letters, and let $\tilde{A}$ be the physical embedding of A by V . There are a nonzero scalar μ , a normal left-canonical bond-two tensor B , and an invertible bond matrix G such that

$$0 < |\mu| < 1, \quad (17)$$

$$\mu B_i = G \tilde{A}_i G^{-1} \quad \text{for every physical pair } i. \quad (18)$$

Adjoin a normal bond-one terminal tensor C , supported on the fifth physical letter, and form the exact weighted direct sum

$$K := (\mu B) \oplus C. \quad (19)$$

The two representatives B, C occur once, with coefficients $\mu, 1$. Their one-site tuples span $M_2(\mathbb{C}) \times \mathbb{C}$, so this presentation is in block-injective canonical form. Both representatives are normal, the coefficient 1 supplies the source normalization, and K is simple. The periodic operators generated by K are positive semidefinite with strictly positive trace at every positive length. Moreover, K saturates the area law and its physical-trace transfer is literally idempotent:

$$\mathcal{E}_K^2 = \mathcal{E}_K. \quad (20)$$

Thus K satisfies the complete standing simple, biCF, MPDO, SAL, ZCL, and normalization conditions of source lines 1628–1665.

For the boundary X in Eq. 9, choose the ambient virtual boundary

$$Y_X := (GXG^{-1}) \oplus 0. \quad (21)$$

The same boundary Y_X is used at every length. Moving the virtual gauge to the boundary and observing that the terminal diagonal block is zero gives

$$\mathcal{K}_N^K(Y_X) = \mathcal{K}_N^{\tilde{A}}(X) \quad \text{for every } N. \quad (22)$$

The physical embedding also gives

$$\mathcal{K}_N^{\tilde{A}}(X) = V^{\otimes N} \mathcal{K}_N^A(X) (V^{\otimes N})^\dagger. \quad (23)$$

Conjugation by $(V^{\otimes N})^\dagger$ recovers every matrix on this range. On the orthogonal complement it can be completed by tracing and preparing a fixed faithful density matrix. This gives a trace-preserving completely positive recovery map on the entire five-letter matrix algebra.

Suppose now that the ambient tensor K admitted the map asserted in Eq. 1. Apply that map to the boundaries Y_X , compose on the input with the physical embedding at length two, and compose on the output with the recovery at length three. Equations 22 and 23 would produce a channel from $\mathcal{K}_2^A(X)$ to $\mathcal{K}_3^A(X)$ for every X . This contradicts Eq. 15. The identical argument at length four rules out the twice-applied conclusion in Eq. 2.

The finite calculations, physical recovery, and complete standing-hypothesis witness are formalized in the accompanying development.¹

4 Verdict

Proposition C.15 is false as printed. The counterexample satisfies every standing hypothesis used in Appendix C.2, including the simple biCF presentation, the MPDO condition, saturation of the area law, literal zero correlation length, common-weight absorption, and the global unit-weight normalization. Nevertheless, no trace-preserving completely positive map can satisfy its two-to-three-site closure equation for every virtual boundary.

The same witness also rules out the twice-applied two-to-four-site map used in the logical implication (ii) $\Rightarrow$ (v) of Theorem 4.9, printed with the cyclic label (iii) $\Rightarrow$ (v) in its closing proof. Conditional channel constructions from supplied neighboring-trace factorizations remain valid, but those extra factorizations are not consequences of SAL and ZCL.

¹The source-facing conclusion is `MPOTensor.CPSVBlockingChannelCounterexample.printed_propositionC15_and_theorem49_ii_to_v_are_false`; its ambient standing-hypothesis witness is `MPOTensor.NeighboringTraceObstructionAmbientCounterexample.exists_ambient_simpleBiCF_neighboringTraceObstruction`.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.