The Projector-Closure Sufficient Condition for Canonical Form

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Verdict: clarification (resolved).

1 The source assertion

Cirac, Pérez-García, Schuch, and Verstraete state a sufficient condition for canonical form in Section II of their analysis of matrix product density operators [CPGSV16]. In the notation of that paper, assume that the tensor has no nontrivial periodic vectors and that every orthogonal projection P satisfying

$$PA^i = PA^iP \qquad \text{for every } i \tag{1}$$

also satisfies

$$A^iP = PA^iP \qquad \text{for every } i. \tag{2}$$

The paper then asserts that the tensor is in canonical form. This is the proposition at arXiv:1606.00608, lines 253–254.

2 The algebraic projection step

The projector hypothesis has a normalization-free consequence. Suppose that P is an orthogonal projection for which

$$(1-P)A^iP = 0 \qquad \text{for every } i. \tag{3}$$

Then

$$(1-P)A^i = (1-P)A^i(1-P). \tag{4}$$

Applying the source hypothesis to $1-P$ gives

$$A^i(1-P) = (1-P)A^i(1-P), \tag{5}$$

and hence both off-diagonal corners vanish. Consequently

$$PA^i = A^i P \qquad \text{for every } i. \tag{6}$$

Thus every projection used in the invariant-subspace decomposition is a reducing projection. This implication is formalized directly, without a left-canonical or unital normalization.[1]

3 The spectral implication

The direct-sum decomposition produces irreducible corner tensors B_k along isometries V_k satisfying $A^i V_k = V_k B_k^i$.[2] The completion of the source proposition consists of two further steps, both now formalized without any normalization hypothesis:[3]

1. every nonzero irreducible corner has a Perron eigenpair $\mathcal{E}_{B_k}(\rho_k) = r_k \rho_k$ with $\rho_k > 0$ and $r_k > 0$; rescaling the corner by $r_k^{-1/2}$ moves its transfer map to spectral radius one and the removed scale into the coefficient $\mu_k := r_k^{1/2}$; and
2. the absence of nontrivial p-periodic vectors, stated for every restriction of A to a minimal invariant subspace, forces the peripheral eigenvalue set of every rescaled corner transfer map to be $\{1\}$, so every rescaled corner is a normal tensor.

No identity of the form

$$\sum_i (A^i)^\dagger A^i = 1 \tag{7}$$

occurs among the hypotheses of the cited proposition, and none is assumed in the formalized completion.

4 Zero blocks and the length-zero coefficient

A corner of A whose matrices all vanish cannot be rescaled to spectral radius one; these are the zero blocks the source allows through the convention $\sum_k D_k \leq D$ stated immediately after its invariant-subspace decomposition

[1] Formal declarations: `MPSTensor.commutes_of_hasInvariantProjectorClosure_of_lowerZero` and `MPSTensor.exists_reducing_projection_of_hasInvariantProj` in `TNLean/MPS/CanonicalForm/ProjectorClosure`.

[2] Formal declaration: `MPSTensor.exists_irreducible_blockDecomp_with_isometry_of_hasInvariantProjectorClosure` in `TNLean/MPS/CanonicalForm/ProjectorClosureDecomposition`.

[3] Formal declarations: `MPSTensor.isNormalTensor_invSqrt_smul_of_unique_peripheral` and `MPSTensor.exists_normalTensor_blockDecomp_sameMPVPos_of_hasInvariantProjectorClosure` in `TNLean/MPS/CanonicalForm/ProjectorClosureSpectral`.

(arXiv:1606.00608, line 219). Dropping a zero block changes exactly one datum of the generated family: the length-zero coefficient, which is the bond dimension. The formalized sufficient condition therefore concludes equality of the generated matrix product vectors at every positive length, together with the source's dimension bound $\sum_k D_k \leq D$. The former all-length variant added the hypothesis that no restriction of A to a minimal invariant subspace is the zero tensor. Since that hypothesis is not part of the source and served only to recover the empty-word coefficient, the variant has been removed. The positive-length formulation is the source-faithful shape of the length-zero bookkeeping analyzed in `docs/paper-gaps/cpsv16_zero_tail_length_zero_decomposition.tex`.

5 Conclusion

The sufficient condition for canonical form at arXiv:1606.00608, lines 253–254, is formalized: projector closure and the absence of nontrivial p-periodic vectors yield a decomposition of the generated matrix product vectors into normal-tensor blocks with nonzero weights, at every positive length and with $\sum_k D_k \leq D$. No separate physical statement is retained for the length-zero coefficient.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.