Power Sums in the BNT Weight Comparison

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Verdict: clarification (historical).

1 The source lemma

This note concerns the scalar power-sum step in the non-periodic Fundamental Theorem proof for matrix product states in canonical form. The source is [CPGSV16, Lemma `Lem:app_simple`, lines 1155–1163]. It considers two finite families

$$(\lambda_{a,k})_{k=1}^{x_a}, \qquad (\lambda_{b,k})_{k=1}^{x_b},$$

ordered by modulus and phase, and assumes

$$\sum_{k=1}^{x_a} \lambda_{a,k}^N = \sum_{k=1}^{x_b} \lambda_{b,k}^N, \qquad N \leq \max\{x_a, x_b\}. \tag{eq:app\_aux}$$

The conclusion is $x_a = x_b$ and equality of the ordered lists. In the equal-case corollary of the Fundamental Theorem, this lemma is applied after the BNT representatives have been matched, to the coefficient families

$$(\mu_{j,q})_{q=1}^{r_{a,j}}, \qquad (\nu_{j,q} e^{i\phi_j})_{q=1}^{r_{b,j}}.$$

The formal statements do not use the source's ordering convention. They state the conclusion as equality of multisets. They do, however, keep the relevant nonzero-coefficient hypothesis: positive powers alone cannot count zero entries.

2 The source-facing formal route

For a fixed BNT representative j, the formal coefficient comparison produces eventual equality of the power sums

$$\sum_{q=1}^{r_j^S} (\mu_{j,q}^S)^N = \sum_{q=1}^{r_j^T} (\mu_{j,q}^T)^N$$

from BNT linear independence. A finite signed sum of nonzero geometric sequences which vanishes eventually vanishes at every exponent, so the equality holds for all positive N. The formal proof then applies the unequal-cardinality finite-range power-sum theorem.[1] The endpoint used by the BNT comparison is the corresponding sector-weight comparison lemma.[2]

The conclusion is

$$r_j^S = r_j^T, \qquad \{\{\mu_{j,q}^S : 1 \le q \le r_j^S\}\} = \{\{\mu_{j,q}^T : 1 \le q \le r_j^T\}\}.$$

This is the multiset form of the source's ordered-list conclusion.

3 The alternative formal route

An earlier formal proof reached the same endpoint by using the recurrence behind finite sums of geometric sequences before applying Newton–Girard. The mathematical input is the following elementary lemma. Let

$$F(N) = \sum_{\ell=1}^{m} c_\ell w_\ell^N, \qquad w_\ell \ne 0.$$

If $F(N) = 0$ for all $N \ge M$, then $F(k) = 0$ for every $k \ge 0$. Indeed, subtracting $w_1 F(N)$ from $F(N+1)$ gives

$$0 = F(N+1) - w_1 F(N) = \sum_{\ell=2}^{m} c_\ell (w_\ell - w_1) w_\ell^N$$

for all $N \ge M$. This is a signed sum with one fewer term. Induction on m shows that the displayed sum is zero for every $N \ge 0$, hence

$$F(N+1) = w_1 F(N) \qquad \text{for all } N \ge 0.$$

Thus $F(k) = w_1^k F(0)$. Since $F(M) = 0$ and $w_1^M \ne 0$, one has $F(0) = 0$, and therefore $F(k) = 0$ for all $k \ge 0$.

For the two sector-weight families, this lemma is applied to the signed union

$$F(N) = \sum_{q=1}^{r_j^S} (\mu_{j,q}^S)^N - \sum_{q=1}^{r_j^T} (\mu_{j,q}^T)^N.$$

The nonzero-weight hypothesis is exactly what permits the step $w_1^M \ne 0$. Since all weights in `SectorWeightData` are nonzero, the exponent-zero power sum is

$$\sum_{q=1}^{r_j} \mu_{j,q}^0 = r_j.$$

[1] `Matrix.sum_pow_eq_implies_card_eq_and_multiset_eq_of_le_max_card`.

[2] `MPSTensor.matched_sector_weight_multiset_eq` in `TNLean/MPS/FundamentalTheorem/SectorBNT/WeightEquiv.lean`, which composes the extension to all positive exponents (`MPSTensor.SectorWeightData.power_sums_eq_of_eventually_eq_hetero`) with the unequal-cardinality power-sum theorem.

Thus $F(0) = 0$ gives $r_j^S = r_j^T$. After transporting the right-hand index set along this equality, the same extrapolation gives equality of the positive power sums over a common index type:

$$\sum_{q=1}^{r_j^S} (\mu_{j,q}^S)^N = \sum_{q=1}^{r_j^S} (\mu_{j,q}^T)^N, \qquad N \geq 1.$$

The proof then applies the same-cardinality Newton–Girard theorem to recover the weight multiset. This route was previously present as checked Lean code through the corresponding copy-count, transported-positive-power-sum, and weight-multiset declarations.[3] Those declarations have been removed from the current source tree because the proportional Fundamental Theorem route no longer uses this same-cardinality detour. The calculation above records the removed argument for comparison with the paper, not a present Lean proof.

4 Conclusion

There is no mathematical gap in the removed alternative route under the nonzero-weight hypotheses used in the former sector-weight data: the extrapolation theorem proves the $N = 0$ identity, so the copy count is justified. Nevertheless, this is no longer a present Lean route. The source-facing theorem surface should use the unequal-cardinality power-sum lemma, because that is the lemma invoked in the paper after BNT representative matching.

There would be a genuine gap if one tried to count zero weights from positive power sums alone. For example, adjoining a zero to a finite family does not change any positive power sum. Hence a statement without a nonzero-coefficient hypothesis can only recover the multiset after deleting zero entries, unless an additional $N = 0$ identity is assumed.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[3] `MPSTensor.SectorWeightData.copies_eq_of_eventually_coeff_eq`; `MPSTensor.SectorWeightData.eventually_coeff_eq_implies_all_pos_eq_after_copies`; `MPSTensor.SectorWeightData.weight_multiset_eq_of_copies_eq_of_coeff_eq`.