

The Missing Algebraic Specification of the PEPS Boundary-RFP Diagrams

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Verdict: open-gap (open).

1 Source assertion

Section 4.2 of Cirac–Pérez-García–Schuch–Verstraete considers a homogeneous square-lattice PEPS tensor A . It assumes the graphical renormalization relation in Figure 62, where the equality is understood up to an isometry on the physical indices. The paper then draws a local tensor for the one-dimensional boundary state and asserts that the PEPS relation produces the two maps in the following diagram, “which gives the desired T and S ” of Definition 4.1 [CPGSV16, lines 694–729].¹

Thus the paper-facing conclusion is that the boundary MPO is a renormalization fixed point. For a virtual operator X , let $M_1(X)$ and $M_2(X)$ denote the one-site and two-site physical matrices obtained by closing the boundary MPO with X . There should be trace-preserving completely positive maps $\mathcal{S}$, from the two-site physical algebra to the one-site algebra, and $\mathcal{T}$, in the reverse direction, such that

$$\mathcal{S}[M_2(X)] = M_1(X), \quad (1)$$

$$\mathcal{T}[M_1(X)] = M_2(X) \quad (2)$$

for every virtual operator X . The existence and channel properties of these maps are conclusions of the PEPS renormalization relation, not additional hypotheses.

2 Boundary data supplied by the cited constructions

The boundary construction cited by the source is algebraic at the level of a finite cylinder. For a cylinder of circumference N_v and bond dimension D , the virtual boundary space has dimension D^{N_v} . After blocking the left and right parts of the cylinder, Ref. [CPSV11, Equation (10)] defines the positive operators

$$\sigma_L = \sum_{I_a} |\widehat{L}^{I_a}\rangle\langle\widehat{L}^{I_a}|, \quad \sigma_R = \sum_{I_b} |\widehat{R}^{I_b}\rangle\langle\widehat{R}^{I_b}|. \quad (3)$$

Equations (11)–(13) of the same reference choose an orthonormal basis of the range of σ_L , define the corresponding support isometry U , and obtain

$$\rho_\ell = U \sqrt{\sigma_L^T} \sigma_R \sqrt{\sigma_L^T} U^\dagger. \quad (4)$$

Thus these equations specify the finite-cylinder transfer space, its two positive boundary operators, and their relation to the reduced density operator.

For a long cylinder, Ref. [SPCPG13, Equation (1) and the following paragraph] writes the boundary state as

$$\sigma \propto \sqrt{\sigma_L^*} \sigma_R \sqrt{\sigma_L^*}, \quad (5)$$

¹Source file: `Papers/1606.00608/MPD0-22-12-17-2.tex:694-729`.

and identifies σ_L and σ_R with eigenvectors of the transfer operator for its largest eigenvalue. The same passage emphasizes that symmetry and topological order can make this fixed point non-unique. These results specify the transfer operators and boundary density operators. They do not define the PEPS blocking isometry in Figure 62 or the two maps in the final renormalization diagram.

3 The local boundary tensor

Under the PEPS renormalization hypothesis introduced at source lines 716–720, the paper identifies the diagram at lines 721–724 with the tensor of the boundary theory. Independently of that identification, the same diagram defines an algebraic ket–bra contraction for every rank-five tensor. Let d be the PEPS physical dimension, and let D_h and D_v be its horizontal and vertical virtual dimensions. Write $A_{l,r,u,b}^s$ for a PEPS tensor with $0 \leq s < d$, $0 \leq l, r < D_h$, and $0 \leq u, b < D_v$, whose virtual legs are ordered left, right, up, and down. Choosing the down leg as the inward leg, the physical leg s and inward virtual leg b are contracted between the ket and bra copies:

$$(E_A)^{u,u'}_{(l,l'),(r,r')} = \sum_{s=0}^{d-1} \sum_{b=0}^{D_v-1} A_{l,r,u,b}^s \overline{A_{l',r',u',b}^s} \quad (6)$$

Consequently, E_A has boundary physical dimension D_v and MPO bond dimension D_h^2 . The source diagram does not label the virtual directions, and directional names are local to each copy. After choosing matching inward legs in the ket and bra copies, we denote their common contracted coordinate by b in both component functions.

The formal definition records this algebraic contraction for an arbitrary tensor. It does not formalize the source’s RFP-dependent identification of the contraction with a boundary-theory tensor.² Moreover, grouping the contracted pair (s, b) into one ancillary coordinate gives a local purification with purifying bond dimension D_h . Thus the contraction E_A is an LPDO and generates positive semidefinite periodic operators at every nonzero length for every A . This is a stronger algebraic positivity statement than the source needs: the paper invokes RFP to identify E_A with the boundary theory, whereas positivity of the displayed Gram contraction itself needs neither a fixed point nor a channel.³

4 Data absent from the diagrams

The source passage does not give an algebraic form of the PEPS blocking relation. In particular, it does not specify the domain and codomain of the physical isometry or the identifications between the external virtual spaces before and after blocking. The cited boundary constructions do specify Eqs. (3)–(5), but they do not select a normalized pair independently of cylinder boundary conditions when the leading transfer eigenspace is degenerate. Nor do they identify such a selection algebraically with the local tensor E_A in Eq. (6).

The refinement part of the final diagram inserts an unlabelled local factor. Its space and normalization are not stated. The coarse-graining and refinement pictures therefore do not by themselves define linear maps on the full one-site and two-site boundary matrix algebras. An algebraic proof of Eqs. (1) and (2) must additionally show that the resulting maps are completely positive and trace-preserving on those full algebras.

These are not merely missing proof steps for a stated algebraic theorem: they are data needed to formulate the theorem. Supplying the channel properties or the normalization as assumptions would yield only a conditional statement and would not formalize the implication asserted by the paper.

5 Formal boundary

The source-faithful local contraction (6) is complete. The transfer operators and finite-cylinder boundary density are specified by Eqs. (3)–(5), but their passage to the local tensor used by the renormalization

²Formal declaration: `TNLean.PEPS.localBoundaryMPOTensor`.

³Formal declarations: `TNLean.PEPS.localBoundaryMPOTensor_isLPDO` and `TNLean.PEPS.localBoundaryMPOTensor_isMPDO`.

diagram is not. The implication from the PEPS renormalization diagram to the boundary MPO renormalization equations therefore remains an open statement-level problem. A faithful completion must first give an algebraic PEPS blocking relation and a boundary fixed-point construction that determine the two channels, and must then derive their channel properties and Eqs. (1)–(2). No additional complete-positivity, trace-preservation, fixed-point normalization, or extension hypothesis may be inserted into the paper-facing implication.

This boundary is tracked in GitHub issue #7371.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.
- [CPSV11] J. Ignacio Cirac, Didier Poilblanc, Norbert Schuch, and Frank Verstraete. Entanglement spectrum and boundary theories with projected entangled-pair states. *Physical Review B*, 83(24):245134, 2011.
- [SPCPG13] Norbert Schuch, Didier Poilblanc, J. Ignacio Cirac, and David Pérez-García. Topological order in PEPS: Transfer operator and boundary hamiltonians. *Physical Review Letters*, 111(9):090501, 2013.