

Blocking and the Parent-Hamiltonian Range in CPSV16

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Verdict: local-correction (resolved).

1 The two lattice statements

Let A be a matrix product state tensor of bond dimension D in the canonical form of Cirac, Pérez-García, Schuch, and Verstraete. Their sharp block-injectivity proposition supplies a basis of normal tensors $A_1, \dots, A_g$ and an integer p such that, writing D_j for the bond dimension of A_j ,

$$1 \leq p \leq 3D^5, \quad \text{span} \left\{ ((A_1^{[p]})^i, \dots, (A_g^{[p]})^i) : i \right\} = \bigoplus_{j=1}^g M_{D_j}(\mathbb{C}). \quad (1)$$

The discussion following Definition 3.9 then states both that A has a parent Hamiltonian of range at most $3D^5$ and that, after blocking, the nearest-neighbour parent Hamiltonian has the expected ground space [CPGSV16, lines 511–527].

The second statement has a direct meaning on the blocked lattice. For every blocked-chain length $N > 2$, it asserts

$$\ker H_2^{(N)}(A^{[p]}) = \text{span} \left\{ V^{(N)}(A_j^{[p]}) : 1 \leq j \leq g \right\}. \quad (2)$$

This is the statement proved in the formal development.¹

The first statement concerns every sufficiently long chain on the original lattice. It asserts that there are BNT representatives $A_1, \dots, A_g$ and a positive integer $L \leq 3D^5$ such that

$$\ker H_L^{(N)}(A) = \text{span} \left\{ V^{(N)}(A_j) : 1 \leq j \leq g \right\}. \quad (3)$$

¹Formal declaration: `MPSTensor.IsCPSVCanonicalForm.exists_bnt_blocked_hasNearestNeighborParentHamiltonian` in `TNLean/MPS/ParentHamiltonian/CPSVBlockedNearestNeighbor`.

For $1 < d$, the formal development proves this all-chain spanning clause simultaneously with $D^2 < d^L$. Thus it also verifies that the local orthogonal-complement interaction is nonzero.²

2 Why the blocked theorem alone is insufficient

A two-site interaction on the blocked lattice acts on $2p$ consecutive sites of the original lattice. Consequently,

$$\text{nearest-neighbour range for } A^{[p]} \implies \text{range } 2p \text{ for the induced interaction on } A, \quad (4)$$

not range p . Moreover, the quantifier $N > 2$ in (2) concerns original chain lengths divisible by p and greater than $2p$. Definition 3.9 instead requires the original-lattice ground-space identity for every chain length strictly larger than the chosen interaction range. Thus the blocked statement alone does not establish (3). In particular, transporting the two-site blocked interaction is not the proof of the first sentence of line 527.

3 Resolution on the original lattice

Choose one normal representative from each BNT phase class and put each representative in a Perron gauge such that

$$\sum_a A_a^j A_a^{j\dagger} = I, \quad \sum_a A_a^{j\dagger} \Lambda_j A_a^j = \Lambda_j, \quad (5)$$

where Λ_j is positive definite. The representative dimensions obey $\sum_j D_j \leq D$, hence $g \leq D$. Quantum Wielandt gives the common block-injectivity length

$$L_0 = D^4. \quad (6)$$

For $g \geq 2$, the proof of [PGVWC07, Theorem 12] gives the BNT kernel equality at range

$$R = 3(g-1)(L_0+1) + 1 \quad (7)$$

whenever $N \geq R + L_0$. Define instead

$$L = 3(g-1)(L_0+1) + L_0 = R + L_0 - 1. \quad (8)$$

Then $N > L$ implies $N \geq R + L_0$. Since $R \leq L$, antitonicity of the cyclic local-constraint spaces and the usual frustration-free inclusion give the sandwich

$$\ker H_L^{(N)} \subseteq \ker H_R^{(N)} = \text{span}\{V^{(N)}(A_j) : 1 \leq j \leq g\} \subseteq \ker H_L^{(N)}. \quad (9)$$

²Formal declaration: `MPSTensor.IsCPSVCanonicalForm.exists_bnt_hasParentHamiltonianGroundSpaceSpanning` in `TNLean/MPS/ParentHamiltonian/CPSVOriginalRange`.

The exact coisometric reconstruction of literal CPSV canonical form preserves the positive-length local space, so this equality transfers from the direct sum of representatives to A . Finally,

$$L \leq 3(D-1)(D^4+1) + D^4 \leq 3D^5. \quad (10)$$

For one representative, one may take $L_{\text{aux}} = D^4 + 1$. The required periodic intersection and closure property for a normal tensor is the result reviewed in [CPGSV21, Section IV.C]. If there are no representatives, one may take $L_{\text{aux}} = 2$; this is a formal boundary extension of the source argument, not an additional claim extracted from CPSV16 or PGVWC07.

To obtain one witness that also satisfies the nontriviality condition, enlarge each auxiliary range to

$$L = 3D^5. \quad (11)$$

Cyclic-range antitonicity preserves the upper inclusion at the enlarged range, and the usual frustration-free inclusion supplies the reverse one. Therefore the spanning equality still holds for every $N > L$. If $D > 0$ and $1 < d$, then

$$D^2 < 2^{2D} \leq d^{2D} \leq d^{3D^5} = d^L. \quad (12)$$

The same witness now satisfies $0 < L$, $D^2 < d^L$, and $L \leq 3D^5$.

4 The physical-dimension boundary

The source first imposes $d^L > D^2$ and then defines the local interaction as the orthogonal complement of the local MPS space. For $D > 0$, this condition is possible for some positive L exactly when $1 < d$: if $d \leq 1$, then $d^L \leq 1 \leq D^2$, whereas the estimate (12) proves sufficiency when $1 < d$. The source-faithful formal theorem therefore states this exact feasibility boundary explicitly and includes $D^2 < d^L$ in its existential conclusion. This is a boundary of the construction, not an extra premise used to prove the ground-space equality. Positive ambient bond dimension is likewise explicit through $D > 0$.

5 Conclusion

The blocked-lattice assertion (2) is formalized with the sharp bound $p \leq 3D^5$. For $1 < d$, the original-lattice assertion (3) is now proved with $0 < L$, $D^2 < d^L$, and $L \leq 3D^5$. The proof absorbs the finite-size window of the source theorem into an auxiliary interaction range, applies the cyclic-range sandwich, and enlarges uniformly to $L = 3D^5$. This resolves the full source construction under its necessary nontrivial physical boundary. The blocked and original-lattice assertions remain distinct: one concerns a nearest-neighbour interaction for the actual blocked tensor, while the other concerns a bounded-range interaction on every sufficiently long original periodic chain.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.
- [PGVWC07] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Information & Computation*, 7(5):401–430, 2007.