

The parent commuting Hamiltonian condition in Cirac–Perez-Garcia–Schuch–Verstraete 2016

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Verdict: scope-restriction (open).

1 Source condition

This Appendix D condition is not the route used in the proof of the forward RFP-to-NNCPH implication. The proof of Theorem 3.10 at source lines 1305–1307 consists of one sentence saying that the implication is immediate from the structural characterization in Theorem 3.11. Definition D.2 belongs to the later, independent discussion of decorrelation.

Definition D.2 of [CPGSV16] states the parent commuting Hamiltonian condition for a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_X \otimes \mathcal{H}_B$. It uses local projectors Q_{AX} and Q_{XB} . Let K_{AX} and K_{XB} denote their kernels, and write

$$\widehat{Q}_{AX} = Q_{AX} \otimes I_B, \quad \widehat{Q}_{XB} = I_A \otimes Q_{XB}$$

for their lifts to the tripartite space. The source condition is¹

$$[\widehat{Q}_{AX}, \widehat{Q}_{XB}] = 0, \tag{1}$$

$$K_{AXB} = (K_{AX} \otimes \mathcal{H}_B) \cap (\mathcal{H}_A \otimes K_{XB}). \tag{2}$$

The proof of Proposition D.3 then writes $P_{AX} = 1 - Q_{AX}$ and $P_{XB} = 1 - Q_{XB}$. Their lifts

$$\widehat{P}_{AX} = P_{AX} \otimes I_B, \quad \widehat{P}_{XB} = I_A \otimes P_{XB}$$

satisfy the product identity

$$\widehat{P}_{AX} \widehat{P}_{XB} = P_{AXB}. \tag{3}$$

This is the identity obtained from commuting orthogonal projectors onto the two local kernels K_{AX} and K_{XB} .² The explicit tripartite declaration **Tripa**

¹Local source: `Papers/1606.00608/MPD0-22-12-17-2.tex:2205–2218`, labels `QAXQXB` and `KAXBKAXKXB`.

²Local source: `Papers/1606.00608/MPD0-22-12-17-2.tex:2259–2289`, in particular labels `PAXQAX`, `PXBQXB`, and `arrggg`.

`titeDecorrelation.CommutingParentHamiltonian` now records the range-intersection equality itself. The equivalence

$$\text{ran } P_{AXB} = \text{ran } \hat{P}_{AX} \cap \text{ran } \hat{P}_{XB} \iff \hat{P}_{AX} \hat{P}_{XB} = P_{AXB}$$

for commuting orthogonal projectors is proved by `TripartiteDecorrelation.hasGroundSpaceIntersection_iff_product_eq`; the forward implication is exposed as `TripartiteDecorrelation.CommutingParentHamiltonian.hproduct`.

2 Current boundary

The current declaration `Decorrelation.HasCommutingParentHam` records only the idempotent-product part of this condition.³ For a complex vector space E , it asks for endomorphisms $P_{AX}, P_{XB} : E \rightarrow E$ such that

$$P_{AX}^2 = P_{AX}, \quad P_{XB}^2 = P_{XB}, \quad P_{AX}P_{XB} = P_{XB}P_{AX}, \quad P_{AX}P_{XB} = P_K. \quad (4)$$

It does not assert a tensor decomposition of E , local action on the AX and XB tensor factors, self-adjointness, orthogonality, or the existence of source projectors Q_{AX} and Q_{XB} . Consequently every idempotent P_K has the tautological choice $P_{AX} = P_{XB} = P_K$, recorded by `Decorrelation.HasCommutingParentHam.ofIdem`.

Thus the present declaration is not Definition D.2 itself. It is the algebraic idempotent-product consequence of Definition D.2 that is sufficient for the backward decorrelation implication already proved in the file.

The nearest-neighbor parent-Hamiltonian development now also has a separate three-site support layer for the AX and XB faces: `MPSTensor.leftPairLift`, `MPSTensor.rightPairLift`, and `MPSTensor.HasOverlappingTwoSiteCommutation`. This records the local action and commutation equation on the common three-site space. The source kernel-intersection equation

$$K_{AXB} = \ker((Q_{AX})_{AX}) \cap \ker((Q_{XB})_{XB})$$

is recorded by `MPSTensor.HasAppendixD2ParentCommutingHamiltonian`, whose forgetful projection gives the algebraic overlapping-commutation predicate. The scalar pair-index equation has now been assembled into the physical map

$$(Uv)_i = \sum_{\alpha, \beta} U_{\alpha, \beta}^i v_{\alpha, \beta}, \quad U^*U = 1, \quad (5)$$

in `MPSTensor.AppendixBStructuralData.physicalIsometry` and `MPSTensor.AppendixBStructuralData.physicalIsometryLeftInverse_comp`. The tensor-power maps now satisfy

$$U^{*\otimes L} U^{\otimes L} = 1 \quad (6)$$

³See `TNLean/MPS/ParentHamiltonian/Decorrelation`.

for every L , as recorded by `MPSTensor.AppendixBStructuralData.physicalIsometryTensorPowerLeftInverse_comp`. On two sites, the inserted virtual bond

$$(I_\varphi v)_{(a,b),(b',c)} = \delta_{b,b'} \lambda_b v_{a,c} \quad (7)$$

has physical image

$$\text{ran}(U^{\otimes 2} I_\varphi) = G_2(A_{\text{core}}). \quad (8)$$

This is `MPSTensor.AppendixBStructuralData.twoSiteBasicEmbedding_range`. The orthogonal projector onto this image is `MPSTensor.AppendixBStructuralData.twoSiteBasicSupportProjection`, and its complementary projector is exactly $q_2(A_{\text{core}})$, as recorded by `MPSTensor.AppendixBStructuralData.twoSiteBasicSupportProjection_eq_complement`. The elementary passage from Q_{AX}, Q_{XB} to their complements $1 - Q_{AX}, 1 - Q_{XB}$ is recorded by `MPSTensor.HasOverlappingTwoSiteCommutation.complement`. The three-site transport from these local projectors to translated parent terms is available in both conventions: directly for Q_{AX}, Q_{XB} through `MPSTensor.localTerm_two_three_zero_one_commute_of_appendixD2`, and for their complements through `MPSTensor.localTerm_two_three_zero_one_commute_of_appendixD2_complement`. For the Appendix B structural datum itself, the current coefficient-space representatives are `MPSTensor.AppendixBStructuralData.appendixBQAXOnCoeffSpace` and `MPSTensor.AppendixBStructuralData.appendixBQXBOnCoeffSpace`; both are identified with the canonical two-site parent interaction $q_2(A)$ before placement on the AX and XB faces. Their factor actions on the homogeneous three-single-site coefficient space are now explicit. Under the canonical two-site and right-associated three-site reindexings,

$$[\hat{Q}_{AX}]_{AX} = [\hat{Q}_{AX}] \otimes I_d, \quad (9)$$

$$[\hat{Q}_{XB}]_{XB} = I_d \otimes [\hat{Q}_{XB}]. \quad (10)$$

The generic matrix identities are `MPSTensor.leftPairLift_toMatrix_reindex` and `MPSTensor.rightPairLift_toMatrix_reindex`; their specialization to the hatted Appendix B representatives is given separately by `MPSTensor.AppendixBStructuralData.appendixBQAX_factor_action` and `MPSTensor.AppendixBStructuralData.appendixBQXB_factor_action`. This realizes only the homogeneous choice $\mathcal{H}_A = \mathcal{H}_X = \mathcal{H}_B = \mathbb{C}^d$ of the factor spaces introduced at source lines 2185–2186. It does not supply the arbitrary factorization used in Definition D.2 at lines 2205–2218. Since $q_2(A)$ is the canonical orthogonal projector onto $G_2(A)^\perp$, these coefficient-space maps are orthogonal. They are not, by themselves, constructions of the source projectors on $\mathcal{H}_A \otimes \mathcal{H}_X$ and $\mathcal{H}_X \otimes \mathcal{H}_B$, as stated in their defining docstrings in `TNLean/MPS/RFP/AppendixBStructuralData.lean`. The additional identification, after transporting the source projectors to coefficient space, is the explicit hypothesis of `MPSTensor.AppendixBStructuralData.hasOverlappingTwoSiteCommutation_of_appendixD2_identified_projectors`. Without that

identification, the lifted commutator of the coefficient-space maps follows directly from the adjacent virtual-bond projectors and their isometric transport, as recorded by `MPSTensor.AppendixBStructuralData.twoSiteBasicSupportProjection_commute_lifts` and `MPSTensor.AppendixBStructuralData.hasOverlappingTwoSiteCommutation`. The corresponding adjacent parent terms commute on every periodic chain with $N > 2$, by `MPSTensor.AppendixBStructuralData.adjacent_twoSite_localTerms_commute`. For an injective tensor, the general three-site MPS intersection identity is now recorded by `MPSTensor.groundSpace_three_eq_adjacent_twoSite_parent_kernels`:

$$G_3(A) = \ker(q_2(A)_{AX}) \cap \ker(q_2(A)_{XB}). \quad (11)$$

Consequently the Appendix B representatives, placed on their two faces, satisfy the coefficient-space analogues of the idempotence, commutation, and kernel-intersection equations in Definition D.2, with $\hat{K}_{AXB} = G_3(A)$, as proved by `MPSTensor.AppendixBStructuralData.hasAppendixD2ParentCommutingHamiltonian`. This theorem does not identify these hatted maps with the source projectors on the two explicitly factored Hilbert spaces. For a normal left-canonical RFP tensor, the injectivity used by the intersection property is derived from the RFP hypotheses in `MPSTensor.rfp_hasAppendixD2ParentCommutingHamiltonian_of_leftCanonical`; it is not an additional assumption.

3 Elimination plan

The general finite-dimensional tripartite equivalence is now recorded by `TripartiteDecorrelation.parentHamiltonian_iff_decorrelated`. Its parent condition uses orthogonal projectors on the explicit factors AX and XB , and records literally that their lifted ranges have the source intersection. The theorem `TripartiteDecorrelation.hasGroundSpaceIntersection_iff_product_eq` proves that this is equivalent to their lifted product being P_{AXB} . The forward direction constructs these projectors as the supports of the two marginals, and the reverse direction derives and uses the product identity. The local correction to the marginal-support expansion at source lines 2236–2252 is documented in `docs/paper-gaps/cpsv16_decorrelation_slice_expansion_fix.tex`.

The coefficient-space analogue of the algebraic and kernel-intersection clauses is now proved: the operators are the canonical orthogonal parent interactions $q_2(A)$, their lifts commute, and their kernel intersection is $G_3(A)$. It remains to construct the source projectors on the two stated tensor factors and identify their transported actions with these coefficient-space maps. The conditional comparison theorem cited above records precisely this missing identification. This remaining comparison belongs to the RFP application, not to the general decorrelation equivalence. In particular, `Decorrelation.HasCommutingParentHam` remains only its algebraic idempotent-product consequence, and the generic `MPSTensor.HasAppendixD2ParentCommutingHamiltonian` structure does

not itself record self-adjointness. The explicit tripartite condition above does record self-adjointness and proves both directions of Proposition D.3.

For Theorem 3.10, the remaining obligations are instead the full canonical-form scope and the all-chain ground-space spanning statement. They are tracked by GitHub issue #2633; Definition D.2 should not be presented as their missing proof route.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.