Historical Note: Non-Dominant Per-Block Projection in CPSV16 §II Step 1

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Verdict: unfaithful (historical).

1 Status

Historical proof archaeology and a forbidden route, not an active gap. The corrected proportional fundamental theorem is `MPSTensor.fundamentalTheorem_proportional_canonicalForm`. It compares active BNT families whose listed copy weights are nonzero and obtains a bijection of the basis sectors with gauge-phase equivalences. The unrestricted statement printed in CPSV16 remains false: its weak BNT convention allows an unrelated normal representative whose coefficient is identically zero, so the claimed equality of BNT cardinalities fails. The analysis below is retained because it explains why non-dominant per-block projection cannot prove the proportional theorem and must not be revived.

2 Notation

Fix two families of MPS site tensors $\{A_j\}_{j=1}^{r_A}$ and $\{B_k\}_{k=1}^{r_B}$ with nonzero scalars $\mu_j^A, \mu_k^B \in \mathbb{C}$. The assembled MPV at length N is

$$V^{(N)}(A_{\mathrm{tot}}) = \sum_{j=1}^{r_A} (\mu_j^A)^N \, V^{(N)}(A_j), \quad V^{(N)}(B_{\mathrm{tot}}) = \sum_{k=1}^{r_B} (\mu_k^B)^N \, V^{(N)}(B_k), \tag{1}$$

written $V^{(N)}(\texttt{toTensorFromBlocks}\mu A)$ in the formalization.[1] A cross-overlap $\langle V^{(N)}(A_j), V^{(N)}(B_k)\rangle$ is `MPSTensor.mpvOverlap`. *Eventual proportionality* of the assembled families is the hypothesis that for some scalar sequence $c_N \neq 0$ cofinally,

$$\sum_j (\mu_j^A)^N V^{(N)}(A_j) = c_N \sum_k (\mu_k^B)^N V^{(N)}(B_k) \qquad (\forall^{\mathrm{cof}} N), \tag{2}$$

[1] `MPSTensor.toTensorFromBlocks`; `MPSTensor.mpv_toTensorFromBlocks_eq_sum`.

formalized as `EventuallyNonzeroProportionalMPV`$_2$.

The retired one-copy normal-canonical BNT route carried a single weight μ_k per sector with the strict ordering

$$|\mu_1| > |\mu_2| > \cdots > |\mu_r|. \tag{mu\_strict\_anti}$$

Together with the dominant normalization $|\mu_1| = 1$, every sub-dominant block had modulus strictly less than 1. These fields are not part of the current `IsNormalCanonicalFormBNT` predicate, which allows equal-modulus blocks. The discussion below concerns the deleted strict-modulus variant. [CPGSV21, Definition 4.2 and lines 1864–1884] states the source structure using repeated copies and raw weights $\mu_{j,q}$ with sector multiplicities $r_j \geq 1$. The current `SectorDecomposition` surface retains those power-sum coefficients; see `docs/paper-gaps/cpsv16_ft_one_copy_scope_restriction.tex`.

3 Cited assertion

The block-selection paragraph in the proof of Theorem `thm1` [CPGSV16, line 1182] fixes any B-block B_{k_0} and rules out

$$\langle V^{(N)}(A_j), V^{(N)}(B_{k_0})\rangle \xrightarrow[N\to\infty]{} 0 \qquad (\text{all } j), \tag{$\mathcal{H}_{k_0}$}$$

by Corollary `Lem1` [CPGSV16, line 1131], which gives eventual linear independence of $\{V^{(N)}(A_j)\}_j \cup \{V^{(N)}(B_{k_0})\}$ from ($\mathcal{H}_{k_0}$). The source then treats this as contradicting the proportionality identity at [CPGSV16, eq. `eq:II_ABasicTensors`, line 286]. This is the fixed-block claim made at line 1182. The passage does not assume all-pairwise cross-family decay and does not establish linear independence of the full combined family.

4 Retired formal target

The historical target was, for $k_0 : \mathrm{Fin}\, r_B$ arbitrary: given the one-copy normal-canonical BNT hypotheses on both μ^A, A and μ^B, B, `EventuallyNonzeroProportionalMPV`$_2$ between the assembled tensors, an index $k_0 : \mathrm{Fin}\, r_B$, and the hypothesis $\forall j, \langle V^{(N)}(A_j), V^{(N)}(B_{k_0})\rangle \to 0$, derive $\perp$. The old one-copy fixed-block declarations for this target have been retired. The corrected `SectorBNT` proportional matching theorem keeps the raw coefficients of `SectorDecomposition` visible and uses the API lemmas in `TNLean/MPS/FundamentalTheorem/SectorBNT/Api`. It does not require a separate universally quantified fixed-block projection theorem.

5 Where the per-block projection fails for non-dominant k_0

Projecting (2) onto $V^{(N)}(B_{k_0})$ gives, with $\mathrm{LHS}_N := \langle V^{(N)}(A_{\mathrm{tot}}), V^{(N)}(B_{k_0})\rangle$ and $\mathrm{RHS}_N := c_N\langle V^{(N)}(B_{\mathrm{tot}}), V^{(N)}(B_{k_0})\rangle$,

$$\mathrm{LHS}_N = \sum_j (\mu_j^A)^N \langle V^{(N)}(A_j), V^{(N)}(B_{k_0})\rangle, \quad \mathrm{RHS}_N = c_N \sum_k (\mu_k^B)^N \langle V^{(N)}(B_k), V^{(N)}(B_{k_0})\rangle. \tag{3}$$

With $|\mu_j^A| \leq 1$, every $(\mu_j^A)^N$ is bounded; together with $(\mathcal{H}_{k_0})$ this forces $\mathrm{LHS}_N \to 0$. For RHS_N, the surviving cross-overlap decay statement `MPSTensor.IsBNTCanonicalForm.cross_overlap_basis_tendsto_zero` gives $\langle V^{(N)}(B_k), V^{(N)}(B_{k_0})\rangle \to 0$ for $k \neq k_0$, while the diagonal block has a nonzero limiting coefficient, so $\mathrm{RHS}_N = c_N(\mu_{k_0}^B)^N(\ell + o(1))$. In the dominant block, the proportionality identity and the BNT normalizations give the dominant-adjusted scalar limit $\left|c_N(\mu_0^B/\mu_0^A)^N\right| \to 1$. Therefore

$$\left|c_N(\mu_{k_0}^B)^N\right| \sim |\mu_0^A|^N \left|\mu_{k_0}^B/\mu_0^B\right|^N. \tag{4}$$

Under $|\mu_0^A| = 1$ and the retired strict ordering, the second factor in (4) decays geometrically whenever $k_0 \neq 0$, so $\mathrm{RHS}_N \to 0$ together with LHS_N. No contradiction is available from the per-block projection. The dominant case $k_0 = 0$ is the unique index for which the ratio in (4) is the constant 1 and the bound stays uniform; that is the analytic content that the dominant fixed-block statement captured before the retirement of the old one-copy layer.

Two independent probes confirm the obstruction. Path α (`docs/audits/2026-05-13_cpsv16_ft_exact_leading_coeff.md`, PR #1654) shows an exact eventual leading-coefficient identity $c_N(\mu_0^B)^N\zeta^N = (\mu_0^A)^N$ fails at $r_A = r_B = 2$ with distinct sub-dominant moduli. Path β scope adjustment (`docs/audits/2026-05-13_cpsv16_ft_path_beta_scout.md`, PR #1664) shows the lifted per-block projection on the `SectorDecomposition` surface inherits the same decay mismatch.

6 The fixed-block source claim and the formal repair

The Step 1 passage at [CPGSV16, line 1182] asserts the fixed-block contradiction described above. Isolating $V^{(N)}(B_{k_0})$ in (2) gives

$$\sum_j (\mu_j^A)^N V^{(N)}(A_j) - c_N \sum_{k \neq k_0} (\mu_k^B)^N V^{(N)}(B_k) = c_N(\mu_{k_0}^B)^N V^{(N)}(B_{k_0}). \tag{5}$$

The single-block linear independence of $\{V^{(N)}(A_j)\}_j \cup \{V^{(N)}(B_{k_0})\}$ does not control the remaining vectors $V^{(N)}(B_k)$ on the left of (5). Thus it does not by itself justify coefficient comparison in the full proportionality identity. The source passage supplies neither all-pairwise cross-family decay nor a full combined-family linear-independence statement.

The formal development replaces this step rather than formalizing it literally. For a chosen P-block it collects the P-blocks whose overlaps with every Q-block decay. The complementary P-blocks already admit scalar relations with matched Q-blocks. It then applies `MPSTensor.restricted_combined_family_eventually_li` to the decaying P-subfamily together with all Q-blocks and compares coefficients in that restricted family. The selected coefficient is forced to vanish eventually, contradicting `MPSTensor.IsBNTCanonicalForm.coeff_not_eventually_zero`. The unrestricted lemma `MPSTensor.IsBNTCanonicalForm.combined_family_eventually_li` is the full-family special case of the same repair. This coefficient-comparison argument proves the current active-sector matching theorem without using the retired per-block projection.

7 Classification

Historical proof archaeology and forbidden-route explanation. The retired dominant fixed-block declarations handled only the leading block on the old one-copy strict-modulus surface. Extending that projection argument to an arbitrary block fails because both sides of the projected proportionality may decay together. This remains a mathematical warning about the method, not an unproved theorem required by the current development.

The source line 1182 fixed-block claim and the current formal repair must be kept distinct. The repair uses restricted combined-family linear independence, scalar relations for the complementary blocks, and the raw power sums $\sum_q \mu_{j,q}^N$ carried by `SectorDecomposition`. Together with `MPSTensor.IsBNTCanonicalForm.coeff_not_eventually_zero`, this gives the active-sector coefficient comparison proved in the current theorem. Collapsing a sector to one strictly ordered weight destroys precisely the information needed for that argument, so non-dominant per-block projection is a route that must not be reintroduced.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.