

The NNCPH ground-state clause in Cirac–Perez-Garcia–Schuch–Verstraete 2016

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Verdict: scope-restriction (open).

1 Source statement

The pure-state main theorem of [CPGSV16] states that, for a tensor A in canonical form generating a family of MPS, the following conditions are equivalent:¹

$$\text{RFP}(A), \quad \text{ZCL}(A), \quad \forall N > 2, |V^{(N)}(A)\rangle \text{ is a ground state of a NNCPH.} \quad (1)$$

The formal implications below use the same range $N > 2$. This range is independent of Beigi’s ordered-cycle formula. At $N = 2$, the two periodic length-two windows traverse the same sites in opposite orders; the sector-graph dimension formula below now treats this convention without extending the CPSV equivalence to length two. The preceding definition constructs

$$H_L^{(N)} = \sum_{j=0}^{N-1} \tau_j(P_L^\perp) \quad (2)$$

and says that a commuting parent Hamiltonian satisfies

$$[\tau_j(P_L), P_L] = 0, \quad j = 1, \dots, L-1. \quad (3)$$

A nearest-neighbor parent Hamiltonian has $L = 2$.² For $L = 2$, write

$$h_i = \tau_i(P_2^\perp) \quad (4)$$

for the translated two-site interaction. Under the periodic translation convention, the source commutation relation $[\tau_1(P_2), P_2] = 0$ is the nearest-neighbor instance of the commutation relation for h_i after passing to orthogonal complements; disjoint translated two-site terms commute by locality.

¹Local source: `Papers/1606.00608/MPD0-22-12-17-2.tex:534--540`, label `thm:main-MPS`.

²Local source: `Papers/1606.00608/MPD0-22-12-17-2.tex:517--524`.

2 Formal boundary

With $h_i = \tau_i(P_2^\perp)$ as above, the current predicate `MPSTensor.IsNNCPH` records the resulting length-two commutation equation

$$h_i h_j = h_j h_i \quad (i, j \in \{0, \dots, N-1\}). \quad (5)$$

The predicate `MPSTensor.IsNNCPHGroundState` adds the following frustration-free ground-vector equation for the same translated length-two interaction:

$$h_i V^{(N)}(A) = 0 \quad (i \in \{0, \dots, N-1\}). \quad (6)$$

Thus it records the annihilation part of the ground-state clause of Theorem 3.10(iii) for the canonical projector parent interaction used in the formal development. The source definition of a parent Hamiltonian also requires the whole ground-state subspace to be spanned by the periodic MPS vectors for $N > L$. That spanning condition is now named separately as `MPSTensor.HasParentHamiltonianGroundSpaceSpanning`; it is not part of `MPSTensor.IsNNCPHGroundState`. The fixed-chain predicate `MPSTensor.HasNNCPHGroundSpace` packages the commutation equation, the zero-energy equation, and the spanning equation

$$\ker H_2^{(N)}(B) = \text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\}. \quad (7)$$

The source theorem should use the all-chain predicate `MPSTensor.HasNNCPHGroundSpaces`, which quantifies this fixed-chain condition over $N > 2$. The predicate is only a named condition; it does not prove that the spanning equation holds for any concrete canonical-form tensor. The equivalence `MPSTensor.hasNNCPHGroundSpaces_iff_forall_isNNCPHGroundState_and_groundSpaceSpanning` records this separation formally: the all-chain source condition is exactly the all-chain ground-vector equations together with the nearest-neighbor instance of the ground-space spanning clause. The easy inclusion in the spanning equation is now recorded: if each component vector $V^{(N)}(A_j)$ is annihilated by the parent Hamiltonian, or more strongly by every local term, then

$$\text{span}\{V^{(N)}(A_j) : j = 1, \dots, g\} \subseteq \ker H_L^{(N)}(B). \quad (8)$$

³ For a block-diagonal tensor with nonzero weights, `MPSTensor.hasParentHamiltonianGroundSpaceSpanning_toTensorFromBlocks_iff_ker_le_bntMPSVectorSpan` now reduces the source spanning equation to exactly the reverse containment, using this easy inclusion. Its nearest-neighbor specialization, `MPSTensor.hasNNCPHGroundSpaces_toTensorFromBlocks_iff_forall_isNNCPH_and_ker_le_bntMPSVectorSpan`, states the all-chain condition as translated two-site commutation together with the same reverse containment for every

³Formal locations: `MPSTensor.bntMPSVectorSpan_le_ker_parentHamiltonian_of_forall_1_annihilates` and `MPSTensor.bntMPSVectorSpan_le_ker_parentHamiltonian_of_forall_frustrationFree`.

$N > 2$. For a direct sum containing one copy of every distinct BNT basis tensor, whole-tensor transfer idempotence now implies this reverse containment. The resulting spanning theorem is `MPSTensor.IsBNTCanonicalForm.rfp_hasParentHamiltonianGroundSpaceSpanning_basisDirectSum`.

The commutation clause is also proved for this multiplicity-one direct sum. The residual matrix entries form one orthonormal family on the disjoint union of the within-sector matrix spaces. The equal-sector virtual bond projector is the sum of the sector bond projectors, extended by zero between unequal sectors. Its two adjacent placements commute sectorwise, while every mixed product vanishes by orthogonality of the canonical summand inclusions. Transport by the common physical isometry identifies this virtual projector with the canonical projector onto the two-site space of the direct sum. Complementing and transporting the resulting three-site relation around the periodic chain proves `MPSTensor.IsBNTCanonicalForm.rfp_implies_nncph_basisDirectSum`. Combining this commutation result with the spanning theorem gives the complete all-chain condition at the same scope, recorded by `MPSTensor.IsBNTCanonicalForm.rfp_implies_hasNNCPHGroundSpaces_basisDirectSum`. This implication uses only BNT canonical form and transfer idempotence of the whole multiplicity-one direct sum. It uses no comparison, crossing, repeated-copy, raw-weight, or externally supplied commutation hypothesis.

This scope-restricted implication does not prove the full source theorem. Together with the reverse implication and the corrected positive-gap zero-correlation-length equivalence, it does prove the three-way equivalence for the multiplicity-one, unit-weight basis direct sum. Repeated copies, arbitrary raw sector weights, adjacent complementary gaps, and the complete unrestricted canonical-form statement remain excluded. For the forward RFP-to-NNCPH direction, the three-site AX/XB support maps are now represented in `MPSTensor.HasOverlappingTwoSiteCommutation`; Appendix B supplies the basic-vector form. The local projectors in Definition D.2 belong to a separate Appendix D discussion and are not invoked by the one-sentence proof of $\text{RFP} \Rightarrow \text{NNCPH}$ at source lines 1305–1307. The coefficients in the basic-vector form now define the physical isometry

$$(Uv)_i = \sum_{\alpha,\beta} U_{\alpha,\beta}^i v_{\alpha,\beta}, \quad U^*U = 1, \quad (9)$$

recorded by `MPSTensor.AppendixBStructuralData.physicalIsometry` and `MPSTensor.AppendixBStructuralData.physicalIsometryLeftInverse_comp`. For every length L , the tensor power $U^{\otimes L}$ and its conjugate coefficient map now satisfy

$$U^{*\otimes L} U^{\otimes L} = 1. \quad (10)$$

This identity is recorded by `MPSTensor.AppendixBStructuralData.physicalIsometryTensorPowerLeftInverse_comp`. The two-site bond insertion

$$(I_\varphi v)_{(a,b),(b',c)} = \delta_{b,b'} \lambda_b v_{a,c} \quad (11)$$

has also been transported through $U^{\otimes 2}$. The range calculation

$$\text{ran}(U^{\otimes 2}I_\varphi) = G_2(A_{\text{core}}) \quad (12)$$

is recorded by `MPSTensor.AppendixBStructuralData.twoSiteBasicEmbedding_range`. The orthogonal projector P_2 onto this range is `MPSTensor.AppendixBStructuralData.twoSiteBasicSupportProjection`; it satisfies

$$P_2 = 1 - q_2(A_{\text{core}}). \quad (13)$$

The complement identity is `MPSTensor.AppendixBStructuralData.twoSiteBasicSupportProjection_eq_complement`. The coefficient-space representatives `MPSTensor.AppendixBStructuralData.appendixBQAXOnCoeffSpace` and `MPSTensor.AppendixBStructuralData.appendixBQXBOnCoeffSpace` are now identified with the canonical two-site parent interaction $q_2(A)$ before placement on the AX and XB faces. Write these coefficient-space maps as $\hat{Q}_{AX}$ and $\hat{Q}_{XB}$. Their two lifts act on the three-site coefficient space; they are not yet identified with the source projectors on $\mathcal{H}_A \otimes \mathcal{H}_X$ and $\mathcal{H}_X \otimes \mathcal{H}_B$. The one-bond virtual projector and its two adjacent placements are now constructed in `MPSTensor.AppendixBStructuralData.virtualBondProjection`, `MPSTensor.AppendixBStructuralData.leftVirtualBondProjection`, and `MPSTensor.AppendixBStructuralData.rightVirtualBondProjection`. Their commutation is transported through $U^{\otimes 3}$ to the two adjacent physical support projectors, giving `MPSTensor.AppendixBStructuralData.twoSiteBasicSupportProjection_commute_lifts` and hence the unconditional coefficient-space relation `MPSTensor.AppendixBStructuralData.hasOverlappingTwoSiteCommutation`. The cyclic three-site restriction transports this commutator to every adjacent pair on every chain with $N > 2$, as recorded by `MPSTensor.localTerm_adjacent_twoSite_commute_of_threeSite_zero_one_commute` and `MPSTensor.AppendixBStructuralData.adjacent_twoSite_localTerms_commute`. The resulting commuting projector family is `MPSTensor.AppendixBStructuralData.hasProductPairLocalProjectors`. The coefficient-space analogues of the Appendix D algebraic and kernel-intersection equations are also now proved locally: `MPSTensor.AppendixBStructuralData.hasAppendixD2ParentCommutingHamiltonian` identifies $\hat{K}_{AXB} = G_3(A)$ with the intersection of the kernels of the lifts of $\hat{Q}_{AX}$ and $\hat{Q}_{XB}$. Its RFP corollary derives injectivity from the normal left-canonical RFP hypotheses. The concrete canonical parent interactions are orthogonal by construction, but the condition recorded by this declaration does not assert self-adjointness. It remains to construct the source projectors and identify their transported actions with the hatted coefficient-space maps. This local coefficient-space result is not used in the chain commutator proof. Consequently the forward commutation and zero-energy statements are proved for normal left-canonical RFP tensors by `MPSTensor.rfp_implies_nncph_of_leftCanonical` and `MPSTensor.rfp_implies_nncph_ground_state_of_leftCanonical`. For a CPSV normal tensor, the spectral-radius-one normalization supplies a pure trace-preserving gauge.

Gauge invariance of the physical RFP relation and of the canonical parent interactions transports the preceding result to `MPSTensor.rfp_implies_nncph` and `MPSTensor.rfp_implies_nncph_ground_state`. Hence these declarations prove the commutation and zero-energy equations for a single CPSV normal tensor without an external axiom. For the same single tensor, transfer idempotence and normality imply one-site injectivity. The injective periodic-chain theorem therefore identifies the kernel of the two-site parent Hamiltonian with the line spanned by the matrix product vector for every $N > 2$. This is recorded by `MPSTensor.rfp_hasParentHamiltonianGroundSpaceSpanning_single`. Combining this equality with the commutation and zero-energy equations gives `MPSTensor.rfp_implies_hasNNCPHGroundSpaces_single`. Thus the full forward NNCPH ground-space condition is proved for one normal sector.

For a BNT canonical form with several distinct basis sectors, transfer idempotence of the whole direct sum gives residual tensors whose one-site entry vectors have identity Gram matrix. Consequently the simultaneous one-site word tuples span the product of all block matrix algebras. This span supplies a common one-letter expression for every block identity and propagates to every positive word length. It replaces the right-canonical hypothesis in the local restriction-intersection calculation. A global one-site change of cut then forces the block boundary matrices to commute with their tensors, so the periodic nearest-neighbor chain space splits into the sum of the single-block chain spaces. Combining this splitting with the single-block injective periodic-chain theorem proves `MPSTensor.IsBNTCanonicalForm.rfp_hasParentHamiltonianGroundSpaceSpanning_basisDirectSum`. The residual-family bond-projector argument independently proves the commutation clause `MPSTensor.IsBNTCanonicalForm.rfp_implies_nncph_basisDirectSum`. Their conjunction is recorded by `MPSTensor.IsBNTCanonicalForm.rfp_implies_hasNNCPHGroundSpaces_basisDirectSum`. This is precisely the multiplicity-one distinct-sector forward case. It does not assert the result for repeated copies or arbitrary raw weights. The reverse theorem requires both the explicit BNT relation `MPSTensor.IsBNT` and the named all-chain condition `MPSTensor.HasNNCPHGroundSpaces`. Under the additional hypotheses that the represented tensor is normal and left-canonical, `MPSTensor.nncph_implies_rfp` proves transfer idempotence without an axiom. These hypotheses select a normal representative and fix its normalization. They are not the full canonical-form hypothesis of Theorem 3.10.

For the multiplicity-one, unit-weight direct sum of all distinct BNT basis sectors, the reverse implication is now proved without the single-sector normality restriction. The theorem `MPSTensor.IsBNTCanonicalForm.isTransferIdempotent_basisDirectSum_of_hasNNCPHGroundSpaces` matches every BNT basis tensor with a distinct normalized Beigi loop tensor. The disjoint loop sectors have zero mixed transfer maps, and this local orthogonality transports through the independent virtual gauge-phase relations. The direct-sum transfer criterion then gives transfer idempotence. This statement still excludes repeated copies and arbitrary raw weights.

The eventual scale-invariance input to Beigi's classification is now proved without this axiom. Let P be a sector decomposition with representatives

$A_1, \dots, A_g$ and total tensor B . If the representative matrix product vectors are eventually linearly independent, the all-chain condition gives

$$\dim_{\mathbb{C}} \ker H_2^{(N)}(B) = g$$

for all sufficiently large N . This is `MPSTensor.HasBNTSectorData.eventually_parentHamiltonian_groundSpace_finrank_eq`. The proof uses only the ground-space spanning equation and the dimension of the span of a finite linearly independent family. It does not use the commuting interaction classification.

The local spatial-decomposition input is also now internal. The canonical two-site parent interaction, written in ordered-pair coordinates as `MPSTensor.twoSiteParentInteractionMatrix`, is Hermitian because it is the transport of an orthogonal projector. If the nearest-neighbor parent terms commute on three sites, their common left-associated coordinates are the two natural overlapping lifts of this matrix. Therefore `MPSTensor.IsNNCPH.twoSiteParentInteractionMatrix_isHermitian_and_overlappingLifts_commute` supplies exactly the hypotheses of Beigi's spatial decomposition, with no canonical-form or ground-space-spanning assumption. Its direct consequence `MPSTensor.IsNNCPH.exists_unitary_blockActions_of_twoSiteParentInteractionMatrix` decomposes the middle physical space into a finite direct sum of left and right tensor factors on which the two adjacent interactions act separately. The symmetric two-sided form is also internal. The complementary two-site interaction is a positive semidefinite orthogonal projector, and its two overlapping lifts commute under the same three-site hypothesis. The theorem `Matrix.exists_positive_etaPair_decomposition_of_overlappingLifts_commute` combines the two one-sided block actions into

$$(U \otimes U)^*(1 - Q_A)(U \otimes U) = \bigoplus_{q,h} \mathbf{1}_{q,l} \otimes \eta_{q,h} \otimes \mathbf{1}_{h,r}.$$

Idempotence of the left-hand side forces every $\eta_{q,h}$ to be idempotent; its positivity then makes it an orthogonal projector. Consequently `MPSTensor.IsNNCPH.nonempty_beigiSectorGraphData` constructs a finite sector graph directly from the three-site commuting-parent hypothesis, and `MPSTensor.IsNNCPH.beigiSectorGraphData` chooses such a graph. No canonical-form, all-chain ground-space, or eventual-dimension hypothesis is used in this construction.

For a finite sector graph, `MPSTensor.BeigiSectorGraphData.parentHamiltonianGroundSpaceES_finrank_eq` proves Beigi's ordered-cycle ground-space dimension formula for every $N \geq 2$. Its period-two specialization `MPSTensor.BeigiSectorGraphData.parentHamiltonianGroundSpaceES_finrank_eq_two` retains the two orientations: if $a \neq b$, then (a, b) and (b, a) are distinct ordered cycles, exactly as stated in Beigi, J. Phys. A 45 (2012) 025306, Section III, equations (2)–(4). No chain-level ground-space equation, eventual degeneracy, or canonical-form datum is assumed in these sector-graph theorems.

At $N = 1$, Beigi's two-site interaction has no specified action. The source defines $P_{j,j+1}$ on sites $j, j+1$ and then separates the actions of $P_{j-1,j}$ and $P_{j,j+1}$

on the middle site; see J. Phys. A 45 (2012) 025306, Section III. When $N = 1$, these site labels all coincide, and the remainder of that section does not add a rule for turning the two-site operator into a one-site operator. CPSV16 does not impose such a rule: Theorem 3.10(iii), local source lines 534–540, only concerns $N > 2$.

The present development uses the project-local convention that an L -site interaction is zero when $L > N$. Accordingly, `MPSTensor.parentHamiltonian_n_two_one_eq_zero` proves $H_2^{(1)} = 0$, and `MPSTensor.parentHamiltonianGroundSpaceES_two_one_eq_top` proves

$$\ker H_2^{(1)} = \mathbb{C}^d. \quad (14)$$

This convention does not extend Beigi’s ordered-cycle equality. An ordered cycle of length one is a self-loop, so the graph expression would instead be

$$\sum_a \dim \operatorname{ran} P_{a,a}. \quad (15)$$

The two-site block decomposition imposes no equality between this sum and d . For example, take

$$A^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (16)$$

Its two-site ground space is spanned by $|01\rangle$ and $|10\rangle$. The corresponding one-dimensional sectors have only the edges $0 \rightarrow 1$ and $1 \rightarrow 0$, so the self-loop sum is zero, whereas the project-local one-site ground space has dimension $d = 2$. Thus the project-local result is recorded separately, while Beigi’s ordered-cycle theorem is proved for every source-defined length $N \geq 2$, including the two opposite orientations at $N = 2$. The resolved source boundary is recorded in GitHub issue #4400: no cited source defines the two-site operator H_1 or a convention for the coincident periodic edge at $N = 1$. GitHub issue #4655 records the separate project-local boundary.

The remaining finite graph comparison in Beigi’s Section IV is now proved without an axiom. For a finite directed graph with nonnegative integral edge weights, the declarations `FiniteWeightedDigraph.cyclicEdgeWeight_eq_one`, `FiniteWeightedDigraph.cycle_eq_constant`, and `FiniteWeightedDigraph.cycleWeightSum_eq_card_loop` show that eventual constancy of the ordered-cycle sum forces every cyclic edge weight to be one, every directed cycle to be a repeated loop, and every positive-length cycle sum to equal the number of positive loops. The proof uses the two repetition maps from consecutive sufficiently large lengths into their common product, exactly as in the source. Applied to the sector datum, `MPSTensor.BeigiSectorGraphData.parentHamiltonianGroundSpaceES_finrank_eq_card_loop` identifies every parent-ground-space dimension with $N > 2$ with the loop number whenever those dimensions are eventually constant. Combining this with the BNT eventual-dimension theorem gives `MPSTensor.BeigiSectorGraphData.parentHami`

`ltonianGroundSpaceES_finrank_eq_card_loop_of_hasBNTSectorData`. This conclusion does not identify the loop states with the original BNT representatives.

The product-of-pairs vector carried by each positive loop is now also constructed explicitly. The declaration `MPSTensor.BeigiSectorGraphData.loopTensor` uses the left factor of the loop sector as its virtual space and denotes the chosen nonzero loop-edge vector by φ_a . At site n , write r_n and l_n for the right and left sector factors, respectively; the cyclic bond pairs r_n with l_{n+1} . For every positive length, `MPSTensor.BeigiSectorGraphData.mpv_loopTensor` identifies its periodic matrix product vector with the sitewise-unitary image of Beigi's cyclic product

$$\prod_n \varphi_a(r_n, l_{n+1}).$$

For a positive loop $a \rightarrow a$, denote the resulting physical state on a chain of length N by $\Psi_a^{(N)}$. This is the construction in Beigi, J. Phys. A 45 (2012) 025306, Section IV, equation (13). The ambient left factor need not be the minimal virtual space: Beigi assumes only that the selected bond vector is nonzero, not that its coefficient matrix has full rank. Writing that matrix as $\Phi_a = X_a Y_a$ through its positive Schmidt rank r_a , the declarations `MPSTensor.BeigiSectorGraphData.minimalLoopCoordinateTensor` and `MPSTensor.BeigiSectorGraphData.minimalLoopTensor` restrict the virtual space to $\mathbb{C}^{r_a}$. Their letters are $(Y_a)_{\alpha,l}$ and $(X_a)_{r,\beta}$, and `MPSTensor.BeigiSectorGraphData.mpv_minimalLoopTensor` proves that this restriction leaves every positive-length periodic vector equal to Beigi's product-of-pairs vector. Exactness of the rank factorization shows that the columns of Y_a and the rows of X_a span $\mathbb{C}^{r_a}$. Their outer products are the letters of the minimal tensor, and hence `MPSTensor.BeigiSectorGraphData.minimalLoopCoordinateTensor_isInjective` proves one-site injectivity. The one-site unitary preserves this spanning property, giving `MPSTensor.BeigiSectorGraphData.minimalLoopTensor_isInjective` and the normality conclusion `MPSTensor.BeigiSectorGraphData.minimalLoopTensor_isNormal`. Here $\|\varphi_a\|_2$ denotes the Euclidean norm of the bond vector, equivalently the Frobenius norm of its coefficient matrix; it is not the supremum norm on the ordinary coordinate-function space. With $P = Y_a Y_a^*$ and $Q = X_a^* X_a$, the transfer map of the minimal coordinate tensor has the rank-one form

$$\mathcal{E}_a(Z) = \text{tr}(QZ)P, \quad \mathcal{E}_a^2 = \|\varphi_a\|_2^2 \mathcal{E}_a.$$

The second identity is formalized by `MPSTensor.BeigiSectorGraphData.transferMap_minimalLoopCoordinateTensor_comp_self`. The declaration `MPSTensor.BeigiSectorGraphData.loopBondVector` chooses a nonzero vector but does not normalize it. Thus transfer idempotence is not asserted for the raw tensor unless $\|\varphi_a\|_2 = 1$. The separate declaration `MPSTensor.BeigiSectorGraphData.normalizedMinimalLoopTensor` defines $\|\varphi_a\|_2^{-1} \tilde{C}_a$ without changing the raw tensor or any of its exact state formulas. Its transfer map is idempotent by `MPSTensor.BeigiSectorGraphData.normalizedMinimalLoopTensor_isTransferIdempotent`, and its length- N matrix product vector

is $\|\varphi_a\|_2^{-N}$ times the original loop state by `MPSTensor.BeigiSectorGraphData.mpv_normalizedMinimalLoopTensor`. No identification with the original basis of normal tensors is asserted at this stage. Consequently, the spanning statement below may equivalently be formulated using the periodic vectors of the minimal Schmidt-support tensors.

For every $N \geq 2$, `MPSTensor.BeigiSectorGraphData.loopProductState_mem_ker_parentHamiltonian` now proves that this vector belongs to the kernel of the length-two periodic parent Hamiltonian. Its Euclidean-space form is recorded by `MPSTensor.BeigiSectorGraphData.loopProductState_mem_parentHamiltonianGroundSpaceES`. The proof uses Beigi’s local edge-ground-space condition in Section III and the positive-loop specialization in Section IV. The restriction $N \geq 2$ is the source-defined range of the two-site periodic parent Hamiltonian used here. As recorded by the resolved clarification in GitHub issue #4400, no source-defined coincident-edge convention extends this statement to $N = 1$. At every positive length the loop vectors are nonzero by `MPSTensor.BeigiSectorGraphData.loopProductStateES_ne_zero`, and distinct loop vectors are orthogonal by `MPSTensor.BeigiSectorGraphData.loopProductStateES_inner_eq_zero_of_ne`. Consequently they are linearly independent, as recorded by `MPSTensor.BeigiSectorGraphData.loopProductStateES_linearIndependent`. If the parent ground-space dimension is eventually constant, membership and the loop-count dimension formula therefore give, for every $N > 2$,

$$\text{span}\{\Psi_a^{(N)} : a \rightarrow a \text{ is a positive loop}\} = \ker H_2^{(N)}(A). \quad (17)$$

This is `MPSTensor.BeigiSectorGraphData.span_loopProductStateES_eq_parentHamiltonianGroundSpaceES`. Its BNT and all-chain specialization is `MPSTensor.BeigiSectorGraphData.span_loopProductStateES_eq_parentHamiltonianGroundSpaceES_of_hasBNTSectorData`. These spanning results alone do not identify the loop states with the original BNT representatives.

For a normal left-canonical representative, that final comparison is now proved. Algebraic normality implies spectral normality, while the normalized minimal loop tensors are both normal and transfer-idempotent. The all-chain ground-space equality places the matrix product vector of the original tensor eventually in the span of the loop-tensor vectors. The normal-tensor overlap dichotomy and the asymptotic orthogonality of distinct loops then show that the original tensor agrees with one loop tensor up to a unit phase and an invertible virtual change of basis. Transfer idempotence is invariant under these transformations. This is the proof of `MPSTensor.nncph_implies_rfp`.

The single-normal result is a scope restriction relative to CPSV16, Theorem 3.10, source lines 534–541 and its proof at line 1307: the formal theorem assumes explicitly that the chosen representative is normal and left-canonical, as well as an explicit BNT relation and the all-chain ground-space condition. It does not state the unrestricted three-way equivalence for an arbitrary tensor in canonical form. In the source canonical decomposition, the normality hypothesis selects the one-sector case $g = 1$. The companion basis-direct-sum theorem now

establishes the $g > 1$ reverse implication at the multiplicity-one, unit-weight representative, using Beigi’s finite-degeneracy classification. Neither statement establishes the unrestricted repeated-copy, raw-weight canonical-form theorem.

The parent-ground-space identity in CPSV16 applies for $N > 2$, whereas the permutation-rigidity lemmas previously required equality of the two matrix-product-vector spans at every length. Only eventual span equality is used in their proofs: one chooses a sufficiently large length at which both families are linearly independent, and all subsequent overlap comparisons are asymptotic. The comparison is therefore now stated under eventual span equality. CPSV16 supplies this hypothesis on the concrete tail $N > 2$, at source lines 522–525 together with lines 1305–1307. The general comparison lemmas assert only the existence of a finite tail: the selected length must also lie beyond the independent linear-independence thresholds, so no common numerical threshold of 2 is asserted. This is a local correction, replacing an unnecessary all-length hypothesis by the finite-tail hypothesis used in the proof and supplied by CPSV16. Once normal representatives have been matched, the resulting gauge–phase relation determines their vectors at every positive length.

The auxiliary declaration `MPSTensor.BeigiSectorGraphData.mem_ker_parentHamiltonian_of_groundBondProduct_mulVec_eq_self` is stated under the sector-data hypothesis, which supplies commutativity of all translated two-site parent interactions. Thus it is a restricted fixed-vector criterion rather than an unrestricted assertion for arbitrary parent interactions. If the criterion is needed independently of Beigi’s decomposition, its hypothesis should be replaced by explicit pairwise commutativity of the translated interactions.

3 Completed representative equivalence and residual scope

The corrected three-way equivalence is now assembled for the multiplicity-one, unit-weight direct sum of the distinct BNT basis tensors:

$$\text{RFP}(A_{\text{basis}}) \iff \text{positive-gap BNT ZCL}(A_{\text{basis}}) \iff \text{HasNNCPHGroundSpaces}(A_{\text{basis}}). \quad (18)$$

The reverse implication is proved both for a normal left-canonical representative and for the multiplicity-one, unit-weight direct sum of all distinct BNT basis sectors. In the latter statement the comparison with Beigi’s locally orthogonal product-of-pairs loop states is performed familywise and requires no axiom. The pairwise equivalences are formalized by `MPSTensor.IsBNTCanonicalForm.isTransferIdempotent_basisDirectSum_iff_hasNNCPHGroundSpaces` and `MPSTensor.IsBNTCanonicalForm.isPositiveGapBNTZCL_basisDirectSum_iff_hasNNCPHGroundSpaces`.⁴

The current proof boundary is therefore: the reverse implication is proved for the corrected basis-direct-sum representative satisfying the source all-chain

⁴Tracked by GitHub issue #2633.

ground-space condition. Repeated copies and arbitrary raw weights remain excluded by the known counterexample; they are not a pending extension of this representative theorem. The printed unrestricted equivalence remains unformalized. The forward commutation implication has the source-unit Appendix B structural datum, its physical isometric embedding and all tensor powers with explicit left inverses, the two-site bond insertion, the equality of its physical image with $G_2(A_{\text{core}})$, its orthogonal support projector, the three-site AX/XB support maps, and the coefficient form of $U^{\otimes L}\varphi_j^{\otimes L}$, the adjacent virtual and physical projectors, their three-site commutator, its transport to every periodic chain with $N > 2$, and the pure trace-preserving gauge passage for a CPSV normal tensor. The even-chain physical-pair factorization remains a separate conditional construction and is not needed for the proved chain commutator. Extending from one normal tensor to the full repeated-sector canonical form is subject to the source obstruction recorded in `docs/paper-gaps/cpsv16_rfp_isometry_scope.tex`. The single-normal-sector ground-space spanning clause and the multiplicity-one distinct-sector extension are proved. Repeated copies and arbitrary raw weights lie outside the corrected theorem. In the reverse direction, eventual ground-space dimension stability, the symmetric sector-graph construction from three-site commuting parent terms, the ordered-cycle dimension formula, the reduction of the sector graph to one-dimensional loops, and the exact matrix product realization and parent-ground-space membership of every positive loop state are proved. The loop vectors are also proved to span the parent ground space for every $N > 2$ under the eventual-dimension hypothesis supplied by BNT eventual linear independence together with the source all-chain ground-space condition. Their comparison with the original tensor is proved for one normal left-canonical representative and familywise for the multiplicity-one, unit-weight BNT basis direct sum; the corresponding transfer-idempotence statements are proved. These representative-level implications are assembled by the two biconditionals above. The residual gap is only the printed unrestricted statement, whose repeated-copy, raw-weight, and adjacent-gap boundaries are excluded from the corrected theorem.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.