

The Positive-Gap Boundary for Distance- and Length-Independent MPDO Correlations

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Verdict: scope-restriction (resolved).

1 Source formula and mixed-state reading

Equation (39) of Ref. [CPGSV16] writes the two-observable contraction of a periodic matrix product vector as

$$\langle V|O_1O_2|V\rangle = \text{tr}(\mathbb{E}_{O_1}\mathbb{E}^{n_2-n_1-1}\mathbb{E}_{O_2}\mathbb{E}^{N+n_1-n_2-1}). \quad (1)$$

Here N is the ring length and $n_1 < n_2$ are the marked sites. Definition 4.2 gives the mixed-state zero-correlation-length condition, and the following sentence states that the corresponding MPDO correlation functions are independent of length [CPGSV16, lines 735–742].

For an MPO tensor M , closing the physical indices gives

$$\mathcal{T}_M = \sum_i M^{ii}, \quad (2)$$

and inserting a one-site observable O in the same contraction gives

$$\mathbb{E}_O = \sum_{i,j} O_{ji} M^{ij}. \quad (3)$$

Thus the mixed-state counterpart of Eq. 1 is

$$C_M(O_1, O_2; g_{\text{mid}}, g_{\text{wrap}}) = \text{tr}(\mathbb{E}_{O_1}\mathcal{T}_M^{g_{\text{mid}}}\mathbb{E}_{O_2}\mathcal{T}_M^{g_{\text{wrap}}}), \quad (4)$$

where

$$g_{\text{mid}} = n_2 - n_1 - 1, \quad (5)$$

$$g_{\text{wrap}} = N + n_1 - n_2 - 1. \quad (6)$$

Equivalently, the ring length is

$$N = g_{\text{mid}} + g_{\text{wrap}} + 2. \quad (7)$$

The contraction in Eq. 4 is unnormalized. The source introduces normalization of finite-chain MPDOs only later, at lines 792–793, for entropic quantities. The unnormalized reading therefore requires no nonvanishing hypothesis absent from Definition 4.2.

2 Positive varied gaps

Definition 4.2 is the literal idempotence equation

$$\mathcal{T}_M^2 = \mathcal{T}_M. \quad (8)$$

It implies

$$\mathcal{T}_M^r = \mathcal{T}_M, \quad r \geq 1. \quad (9)$$

For the length-independence assertion, fix an arbitrary middle gap $g_{\text{mid}} \geq 0$ and compare two positive wrapping gaps $g_{\text{wrap}}, g'_{\text{wrap}} > 0$. Substitution of Eq. 9 for the two wrapping powers gives

$$C_M(O_1, O_2; g_{\text{mid}}, g_{\text{wrap}}) = C_M(O_1, O_2; g_{\text{mid}}, g'_{\text{wrap}}). \quad (10)$$

The middle power is identical on both sides and need not have a positive exponent. By Eq. 7, varying the wrapping gap at fixed middle gap varies the ring length while keeping the observable separation fixed. Thus Eq. 10 is the positive-wrapping-gap scope restriction of the length-independence claim for MPDO correlations at lines 740–742. It includes adjacent marked sites, for which $g_{\text{mid}} = 0$.

For the distance comparison, take two pairs of positive complementary gaps $g_{\text{mid}}, g_{\text{wrap}} > 0$ and $g'_{\text{mid}}, g'_{\text{wrap}} > 0$. Applying Eq. 9 to all four powers gives

$$C_M(O_1, O_2; g_{\text{mid}}, g_{\text{wrap}}) = C_M(O_1, O_2; g'_{\text{mid}}, g'_{\text{wrap}}). \quad (11)$$

If the two gap sums are equal, the ring length in Eq. 7 is fixed and Eq. 11 gives correlation independence under changing the observable distance while the observables are nonadjacent on both complementary arcs. This is the positive-complementary-gap scope restriction of the implication from equation (39) to CID in the discussion at lines 433–500 of Ref. [CPGSV16].

Positivity is required only for an exponent that is replaced using Eq. 9. At a zero gap,

$$\mathcal{T}_M^0 = \mathbb{1}, \quad (12)$$

and Eq. 8 does not identify $\mathbb{1}$ with $\mathcal{T}_M$. Hence the length theorem permits an unchanged zero middle gap but does not compare a zero wrapping gap with a positive one. The distance theorem, which varies both gaps, requires all four compared gaps to be positive. The unrestricted source/CID reading involving a zero varied complementary gap is not claimed.

3 Formal status

The observable insertion in Eq. 3 is `MPOTensor.verticalLoopWith`. The correlation in Eq. 4 is `MPOTensor.periodicTwoPointCorrelation`, and Eq. 10

is `MPOTensor.periodicTwoPointCorrelation_positiveWrapGaps_independent`. This length-independence theorem assumes only literal physical-trace idempotence and positivity of the two compared wrapping gaps; the fixed middle gap may be zero. It introduces no positivity, normalization, nonvanishing, or equal-total-length condition on the MPO tensor.

Eq. 11 is `MPOTensor.periodicTwoPointCorrelation_positiveGaps_independent`. This fixed-ring distance corollary requires positivity of the four compared complementary gaps; equal gap sums give the nonadjacent-on-both-arcs CID specialization.

The scope restriction is resolved by requiring positivity precisely for each gap exponent that is varied. No invariance replacing a zeroth power by a positive power is claimed.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.