# The Sector Decomposition in the GSNNCH Definition of Cirac–Pérez-García–Schuch–Verstraete

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**Verdict:** local-correction (resolved).

## 1 Source definition

Cirac–Pérez-García–Schuch–Verstraete define a density operator on a periodic chain to be a Gibbs state of a nearest-neighbor commuting Hamiltonian when

$$\rho^{(N)} \propto \bigoplus_x n_x \exp\left(-\sum_{j=1}^{N} \tau_j(h^{(x)})\right), \qquad n_x \in \mathbb{N},$$

where the one-site spaces belonging to distinct labels $x$ are orthogonal and $[h^{(x)}, \tau_1(h^{(x)})] = 0$ in arXiv:1606.00608, Definition 4.8, lines 829–842. With the projector-limit convention stated immediately afterward, this is equivalently

$$\rho^{(N)} \propto \bigoplus_x n_x \prod_{j=1}^{N} \tau_j(B^{(x)}), \qquad B^{(x)} \geq 0.$$

The neighboring translates commute; see arXiv:1606.00608, lines 843–850.

The direct sum and the multiplicities are part of the definition. Passing from the exponential form to positive semidefinite bonds does not remove them.

## 2 Present formal boundary

The formalized sector decomposition now records the finite outer sector type, its orthogonal one-site sector projections, the natural multiplicities, and one positive commuting bond in every sector. The finite-chain and tensor-level predicates therefore retain the direct sum and multiplicities of Definition 4.8. The finite-chain predicate requires positivity and unit trace, while the tensor-level predicate applies it to the normalized operator at each length $N \geq 2$.[1]

[1] Formal structure: `MPOTensor.GSNNCHData`. Formal declarations: `MPOTensor.IsGSNNCHAt`, `MPOTensor.IsGSNNCH`, and `MPOTensor.normalizedMPO`.

The auxiliary single-bond presentation is recorded separately: [2]

$$\rho^{(N)} = c_N \prod_{j=1}^{N} \tau_j(B_N), \qquad c_N > 0,$$

without explicit outer sectors. A theorem embeds this presentation as the one-sector case of the source definition. No converse theorem extracting a single bond from arbitrary source sector data is claimed.

The represented finite-chain sector sum now carries its asserted analytic properties: all cyclic translates of one sector bond commute pairwise, the represented operator is positive semidefinite and invariant under the cyclic shift, and a tensor whose finite-chain operators have the sector form with nonzero traces satisfies the full density-operator predicate. In particular, every injective tensor satisfying the saturated area law is now proved GSNNCH through a one-sector witness.[3] The one-sector form is the commuting product form in which Proposition C.8 states its conclusion for an injective tensor.

# 3 Elimination plan

The first part of the original plan—positivity and translation invariance of the represented finite-chain sector sum from the stored sector data—is complete. Appendix C.2 has also been followed through the BNT projector selection formula: the natural multiplicities are retained, positive-length compression proves that each absorbed BNT representative generates MPDOs, the direct-sum entropy identity proves SAL for each representative, and the canonical block equation proves ZCL for each representative. [4] The general outer-sector construction is also complete: pairwise orthogonal one-site projections, supported positive commuting bonds, and the exact sector product identities give the source GSNNCH form with the prescribed natural multiplicities. [5] The support-preserving sector argument is also complete. Every absorbed BNT representative has an exact isometric restriction to the range of its matching projector, the restricted tensor is injective, SAL passes from the ambient representative to this restriction through the sitewise support isometry, and the positive Proposition C.8 bond lifts to a positive two-site operator supported on the tensor square of that projector. The cyclic translates of this lifted bond commute pairwise on every periodic chain of length at least two, and their full periodic product gives

[2]Formal structure: `MPOTensor.CommutingBondProductData`.

[3]Formal declarations: `MPOTensor.GSNNCHData.bondAt_comm`, `MPOTensor.GSNNCHData.unnormalizedState_posSemidef`, `MPOTensor.GSNNCHData.unnormalizedState_submatrix_rotateConfig`, `MPOTensor.isGSNNCH_of_hasGSNNCHForm`, and `MPOTensor.isGSNNCH_of_isInjective_of_isSAL`.

[4]Formal declarations: `MPOTensor.exists_bntProjectorSelection_positiveLength_of_sameMPVâĆĆPos_isSAL`, `MPOTensor.commonWeightAbsorbedBasisMPOTensor_isMPDO_of_sameMPVâĆĆPos_isSAL`, `MPOTensor.commonWeightAbsorbedBasisMPOTensor_isSAL_of_sameMPVâĆĆPos`, and `MPOTensor.commonWeightAbsorbedBasisMPOTensor_isSourceZCL`.

[5]Formal declaration: `MPOTensor.hasGSNNCHForm_of_orthogonalCommutingSectorFamily`.

the ambient finite-chain operator exactly, with normalization scalar one as in the restricted product. [6] Choosing supported products over all absorbed BNT representatives gives an orthogonal commuting sector family once the required one-site supports are available. A source-context corollary now derives pairwise two-sided supports from common copy weights, one-site spanning, the positive neighboring-operator factorizations, and the unstarred layer identity. The local identity is the ordered product

$$\mathcal{K}_y[\beta_y, \alpha_y]\,\mathcal{K}_x[\beta_x, \alpha_x] = 0 \qquad (x \neq y),$$

not an adjoint product. Under the standing biCF hypothesis, each BNT tensor is injective. Equations `AppUkU=rl` and `Appetakhetc` supply positive neighboring operators and hence an MPDO in every sector. Hermiticity then identifies each sector tensor with its physical adjoint through the injective fundamental theorem, so the span of its physical slices is closed under conjugate transpose. The unstarred ordered annihilation identities consequently make the slice-column spaces orthogonal, and their support projections fix every sector slice on both sides. [7]

This formalizes the pairwise support consequence of the earlier Case II data. The ambient factorization `AppUkU=rl`, together with `Appetakhetc`, now supplies the canonical injective active restriction and its positive physical-sector factorization. Its positive commuting bond lifts to a bond $B_x$ on the original one-site space satisfying

$$(P_x \otimes P_x) B_x (P_x \otimes P_x) = B_x.$$

All periodic translates of $B_x$ commute, and their product equals $\rho^{(N)}(\mathcal{K}_x)$ exactly for every $N \geq 2$. Applying this construction to pairwise orthogonal projections gives the orthogonal commuting sector family required for the outer GSNNCH sum. [8]

The active factor supports in the displayed ambient factorization `AppUkU=rl` have now been assembled into the canonical physical restriction. In each active sector the left and right joint column supports are retained; an inactive sector contributes a zero-dimensional coordinate space. The resulting direct-sum inclusion has orthonormal columns, its range is exactly $P_x$, compression has the corresponding sectorwise left–right factorization, and re-embedding recovers $\mathcal{K}_x$ exactly. Thus no identity factor in an unused direction and no complementary

[6] Formal declarations: `MPOTensor.PhysicalSupportRestrictionData`, `MPOTensor.nonempty_physicalSupportRestrictionData_commonWeightAbsorbedBasis`, `MPOTensor.PhysicalSupportRestrictionData.restricted_isSAL`, and `MPOTensor.PhysicalSupportRestrictionData.exists_etaLocalStructureData_lifted_supported`.

[7] Formal declarations: `MPOTensor.IsBNTLayerOrthogonal`, `MPOTensor.PhysicalSectorFactorization.isMPDO_of_neighboringOperator_pos`, `MPOTensor.exists_physicalSlice_conjTranspose_eq_sum_of_isInjective_isMPDO`, `MPOTensor.exists_pairwise_orthogonal_twoSided_physicalSupport`, and `MPOTensor.exists_pairwise_orthogonal_twoSided_physicalSupport_commonWeightAbsorbedBasis`.

[8] Formal declarations: `MPOTensor.PhysicalSectorFactorization.exists_etaLocalStructureData_supported_of_twoSidedPhysicalSlice`, and `MPOTensor.nonempty_orthogonalCommutingSectorFamily_of_ambientPhysicalSectorFactorization`.

zero summand is introduced. This includes the case in which the active family is empty.

Every neighboring operator of this compressed factorization is now identified with the explicit congruence of the corresponding ambient operator from `Appetakhetc`. Positive semidefiniteness follows from congruence. Consequently the canonical restriction has a positive physical-sector factorization indexed precisely by the active ambient sectors, including the vacuous empty-family case.

The canonical joint physical support is contained in every orthogonal projection satisfying the two-sided equations `PjKiPj`. Two-site support monotonicity therefore transports the positive commuting bond from the canonical support to the printed projection $P_x$. Thus the local compatibility obligation between `AppUkU=rl`, `Appetakhetc`, and `PjKiPj` is complete, both for one sector and for a pairwise orthogonal family of sectors.

The source-facing theorem retains the joint five-identity package

`AppKxKy=0`, `AppUkU=rl`, `Appetakhetc`, `Apptralktrrk`, `AppPsiPhi`.

It also states the standing Case II assumptions that the full tensor generates positive semidefinite operators and that the simultaneous one-site tuples of its BNT representatives span the product of their virtual matrix algebras. Every BNT bond dimension is positive. This positivity is not stored by the bare sector-decomposition type and is necessary because algebraic injectivity is vacuous in dimension zero.

The five displayed identities do not imply common copy weights. Appendix C.2 invokes `lemmus` to obtain that equality from zero correlation length, but zero correlation length is absent from Proposition `prop3to4`. The formal proof therefore does not use `lemmus`. At each fixed length $N \geq 2$, sitewise compression by the pairwise orthogonal physical supports isolates

$$q_s(N)\rho^{(N)}(A_s), \qquad q_s(N) = \sum_q \mu_{s,q}^N.$$

Global positivity makes this compression positive. The sector operator is positive and nonzero, so uniqueness of its positive scalar phase shows that $q_s(N)$ is a nonnegative real number. Rescaling the supported commuting bond by

$$\left(\frac{q_s(N)}{n_s}\right)^{1/N}$$

then preserves the natural copy multiplicity $n_s$ and reproduces the exact unnormalized sector term. Since Definition 4.8 chooses its data separately at each length, this proves the proposition without copy independence.

The local Markov labels in Appendix C.2 belong inside one injective BNT component. They are not the outer labels $x$ of Definition 4.8 and must not be substituted for those outer sectors.

# 4 Classification

Verdict: the source GSNNCH data and predicate are formalized, the represented sector sum is proved positive semidefinite and translation invariant with pairwise commuting translates, and the single-bond presentation gives the one-sector case. The exact unstarred BNT layer identity is formalized. Under the standing biCF injectivity and the per-sector MPDO positivity supplied by the positive neighboring-operator factorization, it yields pairwise orthogonal one-site projections supporting every BNT tensor on both sides. This theorem does not assert that the projections sum to the identity. A square-zero family shows that this conclusion would be false for arbitrary MPO tensors without those hypotheses.

The construction from an already supplied orthogonal commuting sector family to the GSNNCH form is available, together with the natural BNT multiplicities. The canonical restriction, its exact sectorwise factorization, and its positive neighboring operators have now been derived from `AppUkU=rl` and `Appetakhetc`, with inactive factor directions removed without a minimality hypothesis. Together with the two-sided supports obtained from `AppKxKy=0`, this gives supported positive commuting bonds with exact periodic products on the printed projections. Fixed-length orthogonal compression proves that the actual BNT power-sum coefficients are nonnegative real numbers, and positive-root bond rescaling absorbs each coefficient while retaining the natural BNT copy number. The source-facing theorem keeps `Apptralktrrk` and `AppPsiPhi` in its statement.

Proposition `prop3to4` is therefore complete under its printed five identities together with the standing Case II assumptions: MPDO positivity, simultaneous one-site spanning of the product of the BNT virtual matrix algebras, and positive BNT bond dimensions. This repairs the proposition by a chainwise argument; it does not validate the printed invocation of Lemma `lemmus` and does not prove copy independence.