Global and Per-Sector Unit-Weight Witnesses in the CPSV16 BNT Argument

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Verdict: scope-restriction (resolved).

1 The source normalization

Cirac–Perez-Garcia–Schuch–Verstraete 2016 normalizes the raw coefficients in the basis-of-normal-tensors (BNT) expansion by a single global condition. Section II.C, line 246 states the condition for the one-index canonical-form coefficients μ_k. The BNT display at lines 287–289 then refines this index into a sector index and a copy index. In the resulting two-layer notation

$$A^i = \bigoplus_{j=1}^{g} \bigoplus_{q=1}^{r_j} \mu_{j,q} X_{j,q} A_j^i X_{j,q}^{-1},$$

the source assumes

$$|\mu_{j,q}| \leq 1 \quad \text{for all } (j,q), \qquad |\mu_{j_*,q_*}| = 1 \quad \text{for some } (j_*,q_*).$$

This is the same normalization, expressed after the two-layer BNT refinement.

Thus the paper-level canonical form does not state that every BNT sector j contains a copy q of modulus one. A sector may contain only strictly subunit copy weights, while some other sector carries the global unit witness.

2 The formal boundary

The formal predicate `MPSTensor.IsBNTCanonicalForm` records the source normalization faithfully: it has a bound on all copy weights and a single global unit-weight witness. It does not contain the per-sector condition

$$\forall j,\ \exists q,\ |\mu_{j,q}| = 1.$$

Several full-basis matching and global-gauge theorems formerly assumed this per-sector condition explicitly. The condition was the price of the *asymptotic* matching route, which applied the Cesaro non-decay lemma sector by sector in the CPSV16 Appendix MPV line-1182 matching argument: a strictly subunit sector coefficient $c_N^{(j)} = \sum_q \mu_{j,q}^N$ tends to zero, so it is invisible to a projection onto a single sector unless that sector carries a unit-modulus copy.

This restriction has since been removed. The exact linear-independence matcher replaces the asymptotic projection: at a fixed sufficiently large length, BNT linear independence detects whether a sector coefficient vanishes, and `MPSTensor.IsBNTCanonicalForm.coeff_not_eventually_zero` shows that $c_N^{(j)}$ is nonzero infinitely often for *any* nonzero copy weights, with no modulus assumption. The matching and global-gauge declarations therefore carry exactly the source hypothesis set (BNT canonical form, with proportional or equal MPVs) and no longer impose the per-sector unit-weight condition.

3 Equal-modulus declaration path

The current formal assembly is organized through `TNLean/MPS/FundamentalTheorem/SectorBNT/`. There is no separate `TNLean/MPS/FundamentalTheorem/Multi/` assembly directory on the current main branch.

The equal-modulus question is logically separate from the per-sector unit-witness restriction above. The current SectorBNT coefficient comparison works with raw sector coefficients

$$c_N^{(j)} = \sum_q \mu_{j,q}^N$$

and the copy weights are recovered from these coefficients by the finite power-sum argument. Equal-modulus copies such as the two copies in $C \oplus (-C)$, and unequal-modulus copies such as $C \oplus (1/2)C$, are both represented by the raw two-layer SectorBNT formulation.

The declaration path is as follows.

- `MPSTensor.IsBNTCanonicalForm` records the raw copy weights $\mu_{j,q}$, their coefficient sums, the bound $|\mu_{j,q}| \leq 1$, and the single global unit witness. It imposes neither a strict ordering of the moduli nor an equal-modulus layer.
- Sectorwise non-decay is supplied, with no modulus assumption, by

  `MPSTensor.IsBNTCanonicalForm.coeff_not_eventually_zero`,

  which shows the raw coefficient sum is nonzero infinitely often for any nonzero copy weights.
- Eventual equality of the sector coefficient sequences is converted into equality of copy weights by `MPSTensor.matched_sector_weight_multiset_eq` and `MPSTensor.matched_sector_weight_equiv`, using the finite power-sum recovery theorem

  `Matrix.sum_pow_eq_implies_card_eq_and_multiset_eq_of_le_max_card`.

- MPSTensor.SectorBNTCopyWeightMatching.of_coeff_identity packages this recovery into the copy-weight matching used by the proportional global-gauge theorem.
- In the equal-MPV path, the declarations

  MPSTensor.ft_sector_bnt_equal_sector_data
  MPSTensor.ft_sector_bnt_equal_global_gauge
  MPSTensor.ft_sector_bnt_equal_mps_gaugeEquiv_literal

  use the same coefficient-to-copy-weight step.

Thus equal-modulus examples such as $C \oplus (-C)$ are handled by the raw coefficient comparison and finite power-sum recovery after sector matching. No theorem in this SectorBNT declaration path assumes a strict ordering of the copy-weight moduli.

The extra hypothesis that once distinguished these statements from the source was not strictness of moduli but the per-sector existence of a unit-modulus copy weight, which is stronger than the single global unit witness stated in CPSV16 line 246. As described in the next section, that hypothesis has been eliminated by the exact linear-independence matcher, so only the single global unit witness remains.

4 How the restriction was eliminated

The restriction is removed by the second of the two strategies above, in its exact form. Rather than projecting asymptotically onto a single sector, the matching works at a fixed sufficiently large length. The P-sectors are split into those whose overlap decays against every Q-sector and the rest; each non-decaying sector is rewritten as an exact scalar multiple of its matched Q-sector, and the residual is a single linear combination over the combined BNT family, which is eventually linearly independent. Linear independence at a fixed length forces every coefficient to vanish, so any putatively unmatched sector would have $c_N^{(j_0)} = 0$ for all large N, contradicting MPSTensor.IsBNTCanonicalForm.coeff_not_eventually_zero. No power of a copy weight is required to have modulus one, and no Cesaro projection or coefficient limit is taken. The proportional case is identical except that the proportionality scalar c_N rescales only the aggregate Q-side coefficients, never the isolated target P-coefficient.

The matcher is MPSTensor.exists_block_match_exact for the equal case and MPSTensor.exists_block_match_exact_of_eventuallyProportional for the proportional case. Consequently the BNT sector-matching declarations are no longer scope-restricted: they carry exactly the source hypothesis set and may be cited as the CPSV16 theorem and corollary on the BNT canonical-form surface.

5 Unitary equal-case refinement

The equal-case conclusion of CPSV16 Corollary C.5 is also complete on this surface. The unitary normalization of a matched sector is applied while retaining the particular phase that appears in the copy-weight identity. Consequently that phase cancels copy by copy, and the repeated sector unitaries assemble into a unitary block-diagonal gauge. The finite permutation restoring the original sector and copy coordinates is itself unitary, so their product is one unitary gauge on the total bond space.

The matched-coordinate statement is `MPSTensor.ft_sector_bnt_equal_unitary_global_gauge_witnesses`; the literal total-tensor statement is `MPSTensor.fundamentalTheorem_equal_canonicalForm_unitary`. Both use only the BNT canonical-form hypotheses and equality of the MPV families. In particular, neither a per-sector unit-weight assumption nor any additional normalization is introduced.

6 Coefficient absorption and normality

The same distinction is load-bearing in the Case-II proof at source lines 1646–1782. If a normal BNT representative is denoted by A_s and its common coefficient is absorbed to form

$$\mathcal{K}_s = \mu_s A_s.$$

For a tensor B, write $\mathcal{E}_B(X) = \sum_i B^i X (B^i)^\dagger$ for its transfer map and $\rho(\mathcal{E}_B)$ for that map's spectral radius. Then

$$\rho(\mathcal{E}_{\mathcal{K}_s}) = |\mu_s|^2 \rho(\mathcal{E}_{A_s}) = |\mu_s|^2.$$

Thus the global unit witness does not imply that every absorbed representative is normal.

This obstruction is kernel-checked by `MPSTensor.spectralRadius_transferMap_smul` and the explicit two-sector, bond-one construction in `TNLean/MPS/MPDO/CaseIIAbsorptionCounterexample.lean`. Its unabsorbed representatives have disjoint doubled-physical support and are normal, its weights are $(1/\sqrt{2}, 1)$, and the ambient physical-trace transfer is literally the identity. The ambient tensor is an MPDO with positive-length trace two, satisfies SAL, has normalized fixed-representative simple canonical form, and its representatives form a biCF family. Nevertheless the first absorbed representative has transfer spectral radius $1/2$ and hence is not `MPSTensor.IsNormalTensor`; see `MPOTensor.CaseIIAbsorptionCounterexample.ambient_isSAL_isSimpleCanonicalForm_and_firstAbsorbed_not_isNormalTensor`.

This proves that the printed absorbed-normality inference is false under all conditions used in the simple Case-II passage, at the normalized fixed-representative boundary. The inference is *proof-path drift*; it does not refute the claimed all-sector conclusion. The remaining assertion is that every absorbed representative admits some physical-sector factorization with positive neighboring operators and normalized rank-one trace coefficients. It does not require the particular inverse-map sectors to have a rank-one trace matrix.

A separate construction in `TNLean/MPS/MPDO/Theorem49RepeatedCopyCounterexample.lean` refutes the literal printed implication. The scalar normal representative $A = 1$ is repeated with raw weights 1 and $1/2$. The theorem `MPOTensor.CaseIIAbsorptionCounterexample.printed_theorem49_iv_to_v_is_false` packages the exact canonical assembly, global unit weight, MPDO and simplicity data, simultaneous one-letter span, and every support, positivity, and factorization clause of condition (iv). Its two-site block nevertheless fails Definition 4.1, since transfer idempotence would force $1+2^{-4} = 1 + 2^{-2}$. Thus (iv)⇒(v) is a refuted *unfaithful statement*. The source (ii)⇒(v) route is distinct: ZCL first makes copy weights common. The non-Cartesian witness in `TNLean/MPS/MPDO/NonCartesianActiveSectorCounterexample.lean` shows that the inherited local analytic properties do not alone imply the printed factorization, but it has no ambient simple-biCF witness. Thus GitHub issue #6775 retains the source-context factorization, while GitHub issue #6793 tracks the alternative direct construction of the blocked channel pair by coarsening or mixing physical-sector labels. The subsequent outer-sector combination is formalized in GitHub issue #6632.

7 Affected declarations

The following declarations *formerly* carried the additional per-sector unit-modulus hypothesis and no longer do:

- `MPSTensor.forall_k_exists_j_nondecaying_overlap_of_sameMPV`,
- `MPSTensor.bijective_match_of_sameMPV`,
- `MPSTensor.ft_sector_bnt_proportional_sector_match_witnesses`,
- `MPSTensor.ft_sector_bnt_proportional_global_gauge_of_copy_weight_matching`,
- `MPSTensor.ft_sector_bnt_proportional_global_gauge_of_coeff_identity`,
- `MPSTensor.ft_sector_bnt_equal_global_gauge`,
- `MPSTensor.ft_sector_bnt_equal_mps_gaugeEquiv_literal`.

The faithful structural normalization remains the global unit-weight field `MPSTensor.IsBNTCanonicalForm.weight_unit_exists` in `TNLean/MPS/FundamentalTheorem/SectorBNT/Basic.lean`; the source assumes only this single global witness, which the exact matcher is now content with.

References