Historical Note: the Fixed-Block Cancellation Step in CPSV16 Theorem II.1

2026-05-11

Verdict: unfaithful (historical).

1 Status

Historical proof archaeology, not an active gap. The corrected proportional fundamental theorem is `MPSTensor.fundamentalTheorem_proportional_canonicalForm`. It compares active BNT families whose listed copy weights are nonzero and obtains a bijection of the basis sectors with gauge-phase equivalences. The unrestricted statement printed in CPSV16 remains false: its weak BNT convention allows an unrelated normal representative whose coefficient is identically zero, so the claimed equality of BNT cardinalities fails. The fixed-block cancellation sentence therefore belongs only to the history of the superseded proof route. The analysis below records why that route was abandoned.

2 Source passage

This note concerns the proof of CPSV16 Theorem `thm1`, Appendix MPV proof line 1182 of [CPGSV16]. The paper fixes a block $k_0 \in \{1, \ldots, r_B\}$ and writes

$$\forall j, \langle V^{(N)}(B_{k_0}), V^{(N)}(A_j)\rangle \xrightarrow[N\to\infty]{} 0 \tag{1}$$

as a contradiction to eventual proportionality of the two full MPV families. Under the BNT decomposition, the proportionality hypothesis is projective proportionality: the scalar c_N is nonzero at the lengths used in the comparison, as recorded in `docs/paper-gaps/cpsv16_nonzero_proportionality_reading.tex`. Thus

$$\sum_j \mu_j^N \, V^{(N)}(A_j) = c_N \sum_k \nu_k^N \, V^{(N)}(B_k), \qquad c_N \neq 0, \tag{2}$$

and isolating the selected block gives

$$\sum_j \mu_j^N \, V^{(N)}(A_j) - c_N \nu_{k_0}^N \, V^{(N)}(B_{k_0}) = c_N \sum_{k\neq k_0} \nu_k^N \, V^{(N)}(B_k). \tag{3}$$

Corollary `Lem1` in [CPGSV16] says that, from the left-hand decay, the family

$$\dim \operatorname{span}\left(\left\{V^{(N)}(A_j)\right\}_{j=1}^{r_A} \cup \left\{V^{(N)}(B_{k_0})\right\}\right) = r_A + 1 \tag{4}$$

for all sufficiently large N. Equivalently, this fixed-block family is linearly independent. Line 1182 treats that conclusion as a contradiction to the full proportionality identity. However, the identity also contains the vectors $V^{(N)}(B_k)$ for $k \neq k_0$, which are not controlled by this single-block linear-independence statement. The source passage does not assert full combined-family independence. [CPGSV16, Appendix MPV proof, line 1182]

3 Formal boundary

The formal boundary in [CPGSV16] is narrower:

$$\forall^{\mathrm{cof}} N, \dim \operatorname{span}\left(\left\{V^{(N)}(A_j)\right\}_{j=1}^{r_A} \cup \left\{V^{(N)}(B_{k_0})\right\}\right) = r_A + 1. \tag{5}$$

The surviving `SectorBNT` API does not formalize the source sentence as a standalone fixed-block contradiction. Instead, the corrected theorem uses a restricted coefficient-comparison argument. It separates the P-blocks whose overlaps with every Q-block decay, represents the complementary blocks by scalar multiples of matched Q-blocks, and applies `MPSTensor.restricted_combined_family_eventually_li` to the decaying subfamily together with all Q-blocks. The full-family lemma `MPSTensor.IsBNTCanonicalForm.combined_family_eventually_li` is the special case in which every cross-family overlap decays. This is a formal repair and replacement for the insufficient fixed-block inference, not a claim about what the source proves at line 1182. The historical target was

$$(\forall j, \langle V^{(N)}(B_{k_0}), V^{(N)}(A_j)\rangle \to 0) \implies \neg\left(\sum_j \mu_j^N\, V^{(N)}(A_j) = c_N \sum_k \nu_k^N\, V^{(N)}(B_k)\right). \tag{6}$$

No current Lean declaration carries the retired fixed-right or fixed-left one-copy fixed-block names. Reintroducing this target would reconstruct an abandoned proof architecture rather than close a current theorem.

The earlier conditional helpers requiring combined-family linear independence (`_phase_sum_li`) and the residual-span variants have been retired in the 2026-05-13 audit; see `docs/audits/2026-05-13_cpsv16_ft_paper_vs_code_structural_map.md`. Those helpers should not be presented as consequences of the fixed-block sentence at Appendix MPV proof line 1182: Corollary `Lem1` supplies only the displayed single-block family. The current restricted-family proof is a separate repair. This mismatch explains why the fixed-block route was retired; it is not a pending obligation on the corrected proportional theorem.

Retirement note: this note intentionally no longer lists the retired no-tail, leading-only, partner-identification, leading-tail, residual-span, or `_phase_sum_li` declarations. They were deleted as wrong-direction or orphan scaffolding in the same audit.

4 Retired discharge plan

The plan recorded in `docs/audits/2026-05-13_cpsv16_ft_sorry_discharge_plan.md` proposed proving an exact eventual leading-coefficient identity

$$c_N \, \mu_0^{B\,N} \, \zeta^N = (\mu_0^A)^N \tag{7}$$

for all sufficiently large N from the proportionality identity and the BNT linear independence on the CPSV fixed-block family. The proposal would then have cloned the equal-MPV tail subtraction and induction architecture without assuming linear independence of all residual blocks on both sides. This plan was superseded by direct coefficient comparison on active BNT families.

4.1 Factored analytic discharge and its lower-bound obstruction

Classification: historical analytic obstruction.

The proposed factored analytic discharge used four explicit analytic hypotheses:

1. every sector weight in both families has unit modulus;
2. off-diagonal cross-overlap decay on the fixed-block family, i.e., $\langle V^{(N)}(B_{k_0}), V^{(N)}(B_k)\rangle \to 0$ for $k \neq k_0$;
3. a nonzero limit of the self-overlap at the fixed block, $\langle V^{(N)}(B_{k_0}), V^{(N)}(B_{k_0})\rangle \to \ell \neq 0$; and
4. an eventual positive lower bound $\|c_N\| \geq \delta > 0$.

Hypothesis (1) was an extra assumption of the retired plan. CPSV16 line 246 states only that every weight has modulus at most one and that at least one weight has unit modulus globally; it does not make every sector weight unitary. The fixed-block overlap and self-overlap inputs in hypotheses (2)–(3) come from the normal-block comparison at the selected block. Hypothesis (4) is also not directly supplied: the cited argument appeals to a dominant-adjusted scalar limit, but the retired plan did not derive the explicit lower bound $\|c_N\| \geq \delta$ from the canonical-form data. The earlier non-cancellation helpers carrying these extra hypotheses belonged to the retired Tier 3 route and should not be cited as present declarations. No current theorem depends on this lower-bound argument.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.