Per-Pair Support in the CPSV16 Figure 11 Decomposition

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Verdict: clarification (resolved).

1 The summed closed-chain identity

Appendix C.4 of Cirac–Pérez-García–Schuch–Verstraete compares two closed-chain expressions for the vertically read two-site tensor B:

$$O_L(B) = \sum_\gamma \frac{m_\gamma^L}{n_\gamma} \operatorname{tr}(\nu_\gamma^L) O_L(M_\gamma) = \sum_{\alpha,\beta} \operatorname{tr}(\mu_\alpha^L) \operatorname{tr}(\mu_\beta^L) O_L(M_\alpha) O_L(M_\beta). \quad (1)$$

Here $L > 0$, and O_L denotes contraction of a closed chain of length L [CPGSV16, Appendix C.4, lines 2011–2020]. The source then introduces the Figure 11 decomposition of the products $M_\alpha M_\beta$ into normal sectors M_γ and positive diagonal multiplicities $\chi_{\alpha,\beta,\gamma}$ [CPGSV16, Appendix C.4, lines 2020–2029].

Equation (1) is an identity only after summing over all ordered pairs (α, β). It does not imply an identity for a fixed pair. Even after the repeated copies within μ_α and μ_β are written separately, an equality of the form

$$O_L(B) = \sum_{p \in \mathcal{P}} O_L(C_p) \quad (2)$$

does not determine any one summand $O_L(C_p)$. A contribution may vanish, and different decompositions of the total sum may redistribute contributions among the pair labels. Linear independence of the vectors generated by the normal tensors separates the final normal label γ after all pair contributions have been collected; it does not recover the pair label p.

Consequently, global equality of closed chains cannot establish that a fixed copy pair has nonzero support, nor can it supply a positive normal corner for that pair. Such a conclusion requires information that retains the pair coordinate before the closing trace is taken.

2 Tensor-level copy-pair corners

Let

$$A^i = \bigoplus_{a \in \mathcal{A}} w_a M_a^i \tag{3}$$

be the retained one-site vertical tensor, where a records both a normal label and one copy of that label. Its product tensor C has a canonical dependent direct-sum decomposition indexed by ordered copy pairs $p = (a, b)$:

$$C^u = \bigoplus_{p \in \mathcal{P}} C_p^u, \qquad C_p^u = w_a w_b (M_a M_b)^u. \tag{4}$$

If E_p denotes the canonical inclusion of the bond space of C_p, then

$$E_p^\dagger E_p = I, \tag{5}$$
$$C^u E_p = E_p C_p^u, \tag{6}$$
$$E_p^\dagger C^u = C_p^u E_p^\dagger, \tag{7}$$
$$C^u = \sum_{p \in \mathcal{P}} E_p C_p^u E_p^\dagger. \tag{8}$$

For distinct pairs $p \neq q$, one also has $E_p^\dagger E_q = 0$. These are tensor-letter identities, not consequences extracted from (1). The range projection $E_p E_p^\dagger$ therefore selects the fixed pair before any word is evaluated or traced.

The vertically read one-site tensor is related to (3) by an exact coisometric reconstruction

$$\widetilde{M}^i = U^\dagger A^i U, \qquad UU^\dagger = I. \tag{9}$$

Squaring this identity gives a coisometry $\widehat{U}$, obtained from $U \otimes U$ by the standard product-coordinate equivalences, such that

$$\widetilde{M^{[2]}}^u = \widehat{U}^\dagger C^u \widehat{U}, \qquad \widehat{U}\widehat{U}^\dagger = I. \tag{10}$$

Thus the squared reconstruction identifies the complete retained product tensor, while the inclusions E_p isolate its copy-pair corners. Together they provide the tensor-level support that is absent from the global closed-chain sum.

Each nonzero C_p admits an exact spectral decomposition into positive multiples of normal tensors:

$$C_p^u = \sum_{k \in K_p} V_{p,k} (\lambda_{p,k} N_{p,k}^u) V_{p,k}^\dagger, \qquad \lambda_{p,k} > 0, \tag{11}$$

where the $V_{p,k}$ are mutually orthogonal isometries. The composites $E_p V_{p,k}$ retain both the pair label and the active normal corner. Composing once more with $\widehat{U}^\dagger$ places the same corner in the bond space of the blocked vertical tensor. Flattening the dependent family $\{(p,k) : p \in \mathcal{P},\ k \in K_p\}$ into a finite label set loses no support information.

The tensor-level reconstruction, pair-corner identities, and flattened positive-length equality are formalized in `TNLean/MPS/MPDO/VerticalProductFusionDecomposition.lean`. [1]

3 Zero copy pairs

The active set K_p in (11) may be empty. There are two relevant cases. First, the bond space of a pair may itself have dimension zero. The canonical inclusion from that empty space remains an isometry, because its isometry equation is an equality of matrices on the empty index type. Second, the pair bond space may be nonzero while every letter of C_p vanishes. Then the entire pair space is a zero corner, and the exact reconstruction is the empty sum

$$C_p^u = 0 = \sum_{k \in \varnothing} V_{p,k}(\lambda_{p,k} N_{p,k}^u) V_{p,k}^\dagger. \tag{12}$$

No positive coefficient and no artificial normal tensor should be assigned to such a pair.

This convention is compatible with the coisometric orientation of the Figure 11 fusion map. A zero pair has an empty active target, and the unique map onto that target is a coisometry. Requiring an isometry on the whole pair bond space would instead impose a full-support hypothesis and would exclude the elementary zero cross-products described in `docs/paper-gaps/cpsv16_figure11_fusion_coisometry.tex`.

For the source multiplicities, an absent normal label is represented by an empty $\chi_{\alpha,\beta,\gamma}$ fiber, not by a spurious positive diagonal entry. Its contribution to every positive-length power sum is zero. This is the Figure 11 instance of the active-only principle for bases of normal tensors: only nonzero canonical corners are covered by the source-facing decomposition.

4 Verdict

The global closed-chain comparison in Appendix C.4 determines the sum of all copy-pair contributions but does not isolate a fixed pair. Per-pair support comes instead from exact squared reconstruction and the canonical orthogonal corners of the retained product tensor. Spectral decomposition is then performed separately on each corner, with an empty active family when the corner is zero. This route neither assumes full support nor introduces normal sectors with identically zero coefficients.

[1] The principal declarations are `MPOTensor.verticalTensor_blockTwo_squared_coisometry_reconstruction`, `MPOTensor.retainedVerticalProductTensor_eq_sum_pairBlocks`, `MPOTensor.RetainedProductSpectralFamily.retained_reconstruction_active`, and `MPOTensor.RetainedProductSpectralFamily.flat_sameMPV`$_2$`Pos`.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.