The Figure 11 Fusion Map Is a Coisometry onto the Active Sectors

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Verdict: local-correction (resolved).

1 The orientation of the fusion map

The canonical-form convention of [CPGSV16] retains only nonzero blocks. If their dimensions are D_k and the original dimension is D, the source states

$$\sum_k D_k \leq D \tag{1}$$

at lines 214–225. Proposition 4.13 uses a matrix U in the orientation

$$U\widetilde{M}U^\dagger = \bigoplus_\alpha \mu_\alpha \otimes M_\alpha \tag{2}$$

(lines 945–951 and 1863–1870). Thus U maps the original space onto the retained direct sum. In the usual terminology it is a coisometry:

$$UU^\dagger = I, \qquad U^\dagger U = P_{\mathrm{act}}, \tag{3}$$

where P_{act} is the projection onto the nonzero canonical sectors. This convention and the necessity of exact reconstruction are established in `docs/paper-gaps/cpgsv17_vertical_isometry_zero_sector.tex`.

Theorem 4.14(iii) has precisely the same orientation. For each pair of BNT labels it states

$$U_{\alpha,\beta}P^{ij}_{\alpha,\beta}U^\dagger_{\alpha,\beta} = D^{ij}_{\alpha,\beta}, \qquad D^{ij}_{\alpha,\beta} = \bigoplus_\gamma \chi_{\alpha,\beta,\gamma} \otimes M^{ij}_\gamma, \tag{4}$$

where $P_{\alpha,\beta} = M_\alpha M_\beta$ (lines 986–993). Appendix C.4 constructs the right-hand side by taking the active canonical blocks of the Figure 11 tensor (lines 2020–2029). Consequently the source-compatible orthogonality relation is

$$U_{\alpha,\beta}U^\dagger_{\alpha,\beta} = I, \tag{5}$$

not $U^\dagger_{\alpha,\beta} U_{\alpha,\beta} = I$ on the whole product bond space.

As in Proposition 4.13, the forward equation alone is insufficient. The active decomposition must also satisfy

$$P^{ij}_{\alpha,\beta} = U^\dagger_{\alpha,\beta} D^{ij}_{\alpha,\beta} U_{\alpha,\beta}. \tag{6}$$

This equation asserts that the omitted common corner is zero. Equations (5) and (6) imply both zipper equations and preserve every nonempty closed word.

2 A zero product sector

Let the physical index have two values, and consider two one-dimensional classical sectors. The first has its only nonzero letter at $(0, 0)$, while the second has its only nonzero letter at $(1, 1)$. Both are nonzero normal tensors. Their product tensor is nevertheless zero, since

$$P^{ij}_{0,1} = \sum_{q=0}^{1} M_0^{iq} \otimes M_1^{qj} = 0 \tag{7}$$

for every i, j: the two factors require respectively $q = 0$ and $q = 1$. This is the cross-sector product in the elementary classical GHZ family.

For this pair every retained multiplicity may have dimension zero. The unique matrix from the one-dimensional product bond space to the empty retained space satisfies (5), (4), and (6). It cannot satisfy $U^\dagger U = I$ on the one-dimensional product space. Hence the reverse orthogonality relation would impose an additional full-support hypothesis not present in Theorem 4.14 or Appendix C.4.

3 Formal realignment

The existing structure `MPOTensor.BNTFusionIsometryFamily` assumes $U^\dagger_{\alpha,\beta} U_{\alpha,\beta} = I$. It remains mathematically valid as a conditional full-support structure, but it is not the unrestricted conclusion of the source theorem. It is retained only for results whose hypotheses explicitly require full support.

The source-facing construction tracked in GitHub issue #4588 uses the active-support family `MPOTensor.BNTFusionCoisometryFamily` with the following data:

1. positive diagonal matrices $\chi_{\alpha,\beta,\gamma}$, allowing zero-dimensional multiplicity spaces;
2. a coisometry $U_{\alpha,\beta}$ from each product bond space onto the retained nonzero direct sum;
3. the forward identity (4);
4. the exact reconstruction identity (6).

The projector-closure canonical decomposition supplies these data directly: the nonzero reducing corners give isometric inclusions into the product bond space, and their adjoints assemble into the required row coisometry. No claim that the active projection is the identity is needed.

The tensor-attached structure `MPOTensor.BNTFusionTensorClause` now uses this same active-support contract: it stores $UU^\dagger = I$, the forward fusion identity, and exact reconstruction. The implication from the renormalization fixed-point maps is proved in `MPOTensor.HasBNTFusionTensorClause.of_isRFPViaTS`, completing the realignment tracked in GitHub issue #4225. Conversion to the stronger `MPOTensor.BNTFusionIsometryFamily` remains permitted only under the additional identity $U^\dagger U = I$, or an equivalent assertion that the product tensor has no discarded zero reducing corner.

Verdict. The phrase "isometries $U_{\alpha,\beta}$" in the source uses the same retained-row orientation as Proposition 4.13. Interpreting it as $U^\dagger U = I$ strengthens the theorem and excludes legitimate zero product sectors. The faithful Figure 11 conclusion is coisometry onto the active direct sum together with exact reconstruction. The source-facing tensor-attached fusion clause and its renormalization-fixed-point construction now implement precisely this conclusion.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.