Finite-Ring Entropy Corrections for CPSV16 Examples 4.10 and 4.11

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Verdict: false-source (open).

1 Scope

Examples 4.10 and 4.11 of Cirac–Perez-Garcia–Schuch–Verstraete [CPGSV16] are printed at lines 897–924 of `Papers/1606.00608/MPDO-22-12-17-2.tex`. This note separates the literal source statements from two finite-ring corrections. The explanatory calculations below use base-two logarithms, as do the numerical values printed in Example 4.10. The formal entropy definition uses natural logarithms; conversion to bits divides every entropy and mutual information by $\ln 2$.

For a normalized four-spin state, let S_L be the entropy of L consecutive spins and set

$$I_L = S_L + S_{4-L} - S_4.$$

Thus

$$I_2 - I_1 = 2S_2 - S_1 - S_3.$$

2 Example 4.10: the correlated bit flip

2.1 Printed channel and intended correction

Each spin consists of a left and a right qubit. The source prints the conjugating operator with the left-qubit label in both tensor factors:

$$\sigma_x^{(n,l)} \otimes \sigma_x^{(n,l)}.$$

This does not describe an operator acting once on each of the two qubits of spin n. The network-consistent correction is

$$U_n = \sigma_x^{(n,l)} \otimes \sigma_x^{(n,r)}, \qquad \mathcal{E}_n(X) = pU_nXU_n^\dagger + (1-p)X.$$

This correction is evidence-backed rather than merely typographical: at $p = 1/4$ it reproduces all four entropy values printed immediately after the channel. No alternative corrected channel is stated in the bundled source.

2.2 Exact four-spin calculation at flip probability one quarter

Start from a periodic ring of four Bell pairs, one on each bond, and apply the corrected channel independently at each spin. The reduced density operators for one through four consecutive spins have the following nonzero spectra, where $a^{[m]}$ denotes an eigenvalue a of multiplicity m:

$$\begin{aligned}
\operatorname{spec}(\rho_1) &= \left\{(1/4)^{[4]}\right\},\\
\operatorname{spec}(\rho_2) &= \left\{(5/32)^{[4]}, (3/32)^{[4]}\right\},\\
\operatorname{spec}(\rho_3) &= \left\{(7/64)^{[4]}, (3/64)^{[12]}\right\},\\
\operatorname{spec}(\rho_4) &= \left\{(41/128)^{[1]}, (15/128)^{[4]}, (9/128)^{[3]}\right\}.
\end{aligned}$$

Their sums are one. Substitution into $S(\rho) = -\sum_\lambda \lambda \log_2 \lambda$ gives

$$S_1 = 2, \qquad S_2 \simeq 2.9544, \qquad S_3 \simeq 3.8802, \qquad S_4 \simeq 2.7839,$$

which are precisely the printed decimal values.

The strict failure of saturation is exact:

$$I_2 - I_1 = \frac{1}{16 \ln 2} \ln\left(\frac{2^{32}7^7}{3^3 5^{20}}\right).$$

Since $\ln 2 > 0$, positivity reduces to the integer comparison

$$2^{32}7^7 = 3537090251849728 > 2574920654296875 = 3^3 5^{20}.$$

Hence $I_2 > I_1$ at $p = 1/4$.

The source also says that $I_2 > I_1$ for every $p \neq 0, 1/2$. This exception list omits $p = 1$: at $p = 1$ every spin is deterministically flipped, so the resulting state is related to the initial Bell-pair ring by an on-site unitary and equality returns. The certified correction is therefore the explicit $p = 1/4$ witness above; the printed universal sentence should not be used as a formal hypothesis.

3 Example 4.11: the literal periodic state

3.1 Exact finite-ring distribution

The source defines diagonal two-qubit bond operators with weight matrix

$$W = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

For a periodic chain of length N, the normalized state is the classical probability distribution

$$\mathbb{P}_N(k_1, \ldots, k_N) = \frac{\prod_{n=1}^{N} W_{k_n, k_{n+1}}}{Z_N}, \qquad k_{N+1} = k_1,$$

where

$$Z_N = \mathrm{tr}(W^N) = \frac{3^N+1}{2^N}.$$

For $1 \le L \le N$, summing the complementary path gives the exact reduced probability

$$\mu_{N,L}(k_1,\dots,k_L) = \frac{\left(\prod_{n=1}^{L-1} W_{k_n,k_{n+1}}\right)(W^{N-L+1})_{k_L,k_1}}{\mathrm{tr}(W^N)}.$$

This formula includes the endpoint cases $L=1$ and $L=N$ and the periodic closing correction; it is not the marginal of an open stationary Markov chain. If $t(k)$ counts the transitions among the $L-1$ internal edges, then this project-derived finite-ring consequence is

$$\mu_{N,L}(k) = \frac{2^{L-t(k)-2}\left(3^{N-L+1}+(-1)^{t(k)}\right)}{3^N+1}.$$

The formal statement uses an equivalent quotient with only nonnegative exponents. Neither block formula is printed in CPSV16.

At $N=4$, the nonzero spectra are

$$\begin{aligned}
\mathrm{spec}(\rho_1) &= \left\{(1/2)^{[2]}\right\},\\
\mathrm{spec}(\rho_2) &= \left\{(14/41)^{[2]}, (13/82)^{[2]}\right\},\\
\mathrm{spec}(\rho_3) &= \left\{(10/41)^{[2]}, (4/41)^{[4]}, (5/82)^{[2]}\right\},\\
\mathrm{spec}(\rho_4) &= \left\{(8/41)^{[2]}, (2/41)^{[12]}, (1/82)^{[2]}\right\}.
\end{aligned}$$

They give

$$I_2 - I_1 = \frac{1}{82\ln 2}\ln\left(\frac{41^{82}5^{50}}{2^{48}13^{52}7^{112}}\right).$$

The exact integer inequality

$$41^{82}5^{50} > 2^{48}13^{52}7^{112}$$

therefore proves $I_2 > I_1$. Numerically the difference is approximately 0.0069396647 bits, but the integer comparison is the proof. Thus the literal four-site periodic state does not saturate the area law, contrary to the printed claim.

3.2 Normalization and likely source of the discrepancy

A global normalization cannot repair the claim. Replacing W by cW for $c>0$ multiplies every unnormalized length-N configuration weight and Z_N by the same factor c^N. The normalized state, all reduced spectra, and $I_2 - I_1$ are unchanged. In particular, replacing W by $(2/3)W$ does not restore saturation.

There is no statement in the cited source identifying the reason for the failed finite-ring claim. A likely explanation is that the periodic state was confused with the open or infinite stationary Markov chain having transition matrix $(2/3)W$ and uniform stationary distribution. Define the binary entropy by

$$h_2(q) = -q \log_2 q - (1-q)\log_2(1-q).$$

For that different process, the entropy of an open block is

$$H_L = 1 + (L-1)h_2(1/3),$$

so it is affine in L. The periodic closing factor $(W^{N-L+1})_{k_L,k_1}$ is absent there. This Markov-chain interpretation is an explanation suggested by the calculation, not a fact asserted by the paper.

4 Formalization boundary

Example 4.10 supplies an exact corrected witness at $p = 1/4$, but the repeated left-qubit label and the incomplete universal exception list must not be formalized as printed. For Example 4.11, the printed saturation clause is false for the literal finite periodic family, while the independent claim that the family does not have zero correlation length is verified both for the literal diagram in [CPGSV16, Definition 4.2, lines 735–739] and for its scale-invariant repair.

The scalar endpoints are now certified in `TNLean/MPS/MPDO/CPSVExamples410411Arithmetic`. The declarations `CPSVExamples410411Arithmetic.example410_integer_inequality` and `CPSVExamples410411Arithmetic.example411_integer_inequality` prove the two exact natural-number comparisons. The declarations `CPSVExamples410411Arithmetic.example410_log_ratio_pos` and `CPSVExamples410411Arithmetic.example411_log_ratio_pos` prove positivity of the corresponding logarithmic ratios. These results entered the project in commit `a79bfc93314f3f721abecdbc184065497672d7e5`.

Project derivation (effective-binary finite-ring blocks). The module `TNLean/MPS/MPDO/CPSVExample411BinarySupport` now proves the exact internal edge product in `MPOTensor.CPSVExample411BinarySupport.internalEdgeProduct_eq`, every entry of W^m in `MPOTensor.CPSVExample411BinarySupport.weightMatrix_pow_apply`, and the complementary path-sum identity in `MPOTensor.CPSVExample411BinarySupport.sum_complementPathWeight_eq_weightMatrix_pow`. The declaration `MPOTensor.CPSVExample411BinarySupport.reducedBlockState_apply_eq_internal_mul_pow` identifies the canonical reduced state with the normalized path formula, and `MPOTensor.CPSVExample411BinarySupport.reducedBlockState_apply` gives the closed diagonal block formula for $1 \le L \le N$. These are finite-ring consequences of the printed neighboring operators, not claims attributed to CPSV16. The result remains entirely in the effective binary model.

Local fix (ambient support inclusion). For Example 4.11, the effective binary one-site space is included in the source-facing two-qubit site by $k \mapsto |k, k\rangle$; it is not identified with the four-dimensional site by an equivalence. The module `TNLean/MPS/MPDO/CPSVExample411Ambient` proves the exact supported one-site slice and vanishing whenever either local physical leg is unsupported. The declarations `MPOTensor.CPSVExample411Ambient.mpo_ambientM_eq_singleKrausMap` and `MPOTensor.CPSVExample411Ambient.mpo_ambientM_supported_apply` give the sitewise periodic-operator congruence and supported positive-length entries. The declaration `MPOTensor.CPSVExample411Ambient.trace_mpo_ambientM_closed_form` gives the ambient trace $Z_N = (3^N + 1)/2^N$.

Project derivation (source zero-correlation-length obstruction). The module `TNLean/MPS/MPDO/CPSVExample411SourceZCL` records the exact two-site constant-zero entries

$$\rho_2^{(4)}(00, 00) = \frac{14}{41}, \qquad \rho_2^{(3)}(00, 00) = \frac{5}{14}$$

for the effective binary model. The declaration `MPOTensor.CPSVExample411Ambient.reducedBlockState_supported_apply` transports these values to the corresponding supported entries of the ambient two-qubit-site marginals. The source-facing zero-correlation relation implies that the normalized two-site marginal obtained from a ring of length four must equal that obtained from a ring of length three. Since $14/41 \neq 5/14$, the effective and ambient tensors both fail that relation. Write

$$\mathcal{T}_M := \sum_i M^{ii}, \qquad \mathcal{T}_{\widetilde{M}} := \sum_i \widetilde{M}^{ii}$$

for their respective physical-trace transfers. The declarations `MPOTensor.CPSVExample411BinarySupport.physTraceTransfer_M_ne_zero` and `MPOTensor.CPSVExample411Ambient.physTraceTransfer_ambientM_ne_zero` use the exact one-site traces

$$\mathrm{tr}(\mathrm{mpo}(M, 1)) = \mathrm{tr}(\mathrm{mpo}(\widetilde{M}, 1)) = 2$$

to show that both physical-trace transfers are nonzero. The declarations `MPOTensor.CPSVExample411BinarySupport.physTraceTransfer_M_not_idempotent` and `MPOTensor.CPSVExample411Ambient.physTraceTransfer_ambientM_not_idempotent` then prove

$$\mathcal{T}_M^2 \neq \mathcal{T}_M, \qquad \mathcal{T}_{\widetilde{M}}^2 \neq \mathcal{T}_{\widetilde{M}}.$$

Thus the fixed effective and ambient tensors themselves fail the literal diagram in [CPGSV16, Definition 4.2, lines 735–739]. The argument does not use the saturation obstruction, entropy, or spectral calculations.

The scale-invariant statement uses the nonzero up-to-positive-scalar relation for the physical-trace transfer, while the literal source diagram fixes the scalar to one. Both are distinct from the legacy doubled-index transfer-map idempotence convention. No conclusion about that legacy convention is made.

Project derivation (exact length-four spectra). The module `TNLean/MPS/MPDO/CPSVExample411Spectrum` specializes the canonical block formula to $N = 4$. It proves the effective nonzero root multisets

$$\{(1/2)^{[2]}\},\quad \{(14/41)^{[2]},(13/82)^{[2]}\},\quad \{(10/41)^{[2]},(4/41)^{[4]},(5/82)^{[2]}\},\quad \{(8/41)^{[2]},(2/41)^{[12]},(1/82)^{[2]}\}$$

for block lengths one through four. Here $a^{[m]}$ denotes multiplicity m. The generic isometric characteristic-polynomial identity `Matrix.singleKrausMap_charpoly_eq_X_pow_mul_of_isometry`, specialized by `MPOTensor.CPSVExample411Spectrum.ambient_charpoly_eq_X_pow_mul_effective`, is

$$\chi_{\widetilde{\rho}_L}(X) = X^{4^L-2^L}\chi_{\rho_L}(X).$$

The ambient two-qubit block has dimension 4^L, whereas the effective binary block has dimension 2^L. Their difference is therefore the multiplicity of the added zero root. Hence the full ambient root multisets have zero multiplicities 2, 12, 56, and 240, respectively. These spectral formulas are finite-periodic project derivations, not statements attributed to CPSV16.

Project derivation (entropy and saturation obstruction). The module `TNLean/MPS/MPDO/CPSVExample411Entropy` first proves that the effective binary tensor is an MPDO: every positive-length periodic operator is diagonal with non-negative cycle weights. It then evaluates the four block entropies directly from the established root multisets. These formal entropies use natural logarithms. In particular,

$$I_2 - I_1 = \frac{1}{82}\ln\left(\frac{41^{82}5^{50}}{2^{48}13^{52}7^{112}}\right) > 0.$$

The bit-valued formula earlier in this note is obtained by dividing by $\ln 2$. The strict inequality reuses `CPSVExamples410411Arithmetic.example411_log_ratio_pos`; neither the spectra nor the integer inequality is recomputed. Therefore the effective binary tensor does not satisfy SAL. Contraposition of SAL descent through the physical isometry gives the same conclusion for the ambient tensor `MPOTensor.CPSVExample411Ambient.ambientM`.

Scope restriction (remaining consequences). The ambient operator congruence, trace equality, exact $N = 4$ reduced-state spectra, entropy identities, finite-ring saturation obstruction, and source-facing zero-correlation-length obstruction for Example 4.11 are complete. The printed independent non-zero-correlation-length clause is thus verified for the literal finite periodic family, while its saturation clause remains false. No conclusion is made about the legacy doubled-index zero-correlation convention or about thermodynamic behavior. Example 4.10's tensor and final entropy links remain open in GitHub issue #6301.

Example 4.12 is outside the entropy correction in this note. Its statement, dependencies, and separate normalization question are unchanged.

5 Verdict

Verdict for Example 4.10: source typo and scope correction. The repeated left-qubit label is replaced by the left–right correlated flip. The corrected $p = 1/4$ calculation gives the claimed separation, while the printed universal exception list must also exclude $p = 1$.

Verdict for Example 4.11: the printed separation is only half correct for the literal periodic state. The exact finite-marginal mismatch verifies the independent claim that the state does not have source-facing zero correlation length. The four-site calculation gives $I_2 > I_1$, so the printed saturation claim fails. Global normalization cannot repair that failure. The suggested open or infinite Markov-chain explanation is not part of the source statement.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.