CPSV16 Lemma `equalMPS` Gauge-Phase Components and Resolution

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Verdict: unfaithful (resolved).

1 The source statement

Cirac–Pérez-García–Schuch–Verstraete 2016 establishes the following structural lemma about overlaps of normal matrix product vectors [CPGSV16, Lemma `equalMPS`, line 1081]:

> Given two NMPV $V_{a,b}$ generated by two NT with corresponding $D_\alpha \times D_\alpha$ matrices A^i_α ($\alpha = a, b$):
>
> 1. $\lim_{N\to\infty} \langle V_\alpha \,|\, V_\alpha \rangle = 1$,
> 2. $\lim_{N\to\infty} |\langle V_b \,|\, V_a \rangle| = 0$ or 1.
>
> In the latter case, $D_a = D_b$ and there exists a non-singular matrix X and a phase ϕ such that $A^i_b = e^{i\phi} X A^i_a X^{-1}$.

The hypothesis is just that the two tensors are NT. The conclusion has three pieces: convergence of the self-overlap, the $0/1$ dichotomy, and a *structural gauge recovery* in the 1 case. No proportionality hypothesis on the MPVs is required.

The proof in [CPGSV16, lines 1093–1117] goes through the cross-transfer-matrix

$$\mathbb{E}_{ab} = \sum_{i=1}^{d} A^i_a \otimes \overline{A^i_b}, \tag{1}$$

analysing its eigenvalue equation $\sum_i (A^i_b)^\dagger X A^i_a = \lambda X$ together with the trace-preserving relation and the strictly-positive spectral fixed point Λ_a. Cauchy–Schwarz applied to the resulting trace inequality forces $|\lambda| < 1$ or $|\lambda| = 1$, and the latter case yields $A^i_b X = \alpha X A^i_a$ with $|\alpha| = 1$ and X an isometry, which after a dimension argument gives the structural conclusion.

2 The formalized same-dimension component

The formalization represents the same-bond-dimension part of the 1 case by separating two mathematical steps. First, the spectral-radius theorem shows that asymptotically unit-modulus overlap forces the cross-transfer spectral radius to be at least 1.[1] This is the contrapositive of the already formalized decay theorem for mixed transfer spectral radius < 1.

Second, the modulus-one spectral theorem turns the modulus-one spectral situation into gauge-phase equivalence by the Cauchy–Schwarz argument with the positive fixed point.[2] The resulting same-dimension gauge-phase theorem has the source's overlap hypothesis and no MPV proportionality hypothesis, but it is stated for two tensors with the same bond dimension.[3] In this same-dimension component the formal theorem changes the normal-tensor hypothesis set in a controlled way: in CPSV16 a normal tensor has no non-trivial invariant projector, and its associated completely positive map has a unique peripheral eigenvalue, equal to its spectral radius and normalized to one. The spectral-radius and Cauchy–Schwarz argument used here drops this peripheral-simplicity assumption and takes irreducibility together with trace-preserving normalization as explicit hypotheses. This is a generalization of the same-bond-dimension gauge step, not a formalization of the rectangular dimension conclusion.

The proportionality-based theorem in the same file is a correct corollary for a stronger hypothesis. It is not a formalization of Lemma `equalMPS`, because the source does not assume proportional MPV families.

3 Formalized rectangular component

Lemma `equalMPS` also concludes $D_a = D_b$ in the unit-overlap case. This is a rectangular statement: before the conclusion is known, the two normal tensors may have different bond dimensions. The proof derives an isometry between the two bond spaces and then uses normality of the larger tensor to rule out a proper missing subspace, forcing equality of dimensions [CPGSV16, lines 1115–1117].

This bond-dimension matching is obtained from the rectangular overlap-decay theorem `MPSTensor.mpvOverlap_tendsto_zero_of_dim_ne_of_irreducible_TP` in `TNLean/Spectral/TransferOperatorGapNT.lean`: if the bond dimensions differ, the overlap tends to zero, contradicting the unit-modulus overlap hypothesis, so a non-decaying overlap forces $D_a = D_b$. The contrapositive is recorded as `MPSTensor.dim_eq_of_mpvOverlap_norm_tendsto_one_of_irreducible_TP` in `TNLean/MPS/FundamentalTheorem/Proportion`

[1] This theorem is `MPSTensor.mixedTransferSpectralRadius_ge_one_of_mpvOverlap_norm_tendsto_one`; it is in `TNLean/MPS/FundamentalTheorem/Proportional.lean` and formalizes a step in the proof of Lemma `equalMPS`, [CPGSV16, lines 1093–1117].

[2] This theorem is `MPSTensor.modulus_one_eigenvalue_implies_gauge_of_irreducible_TP`; it is in `TNLean/MPS/FundamentalTheorem/Proportional.lean`.

[3] The theorem is `MPSTensor.gaugePhaseEquiv_of_overlap_norm_tendsto_one_of_irreducible_TP`; it is in `TNLean/MPS/FundamentalTheorem/Proportional.lean`.

`al.lean`. Its hypotheses and proof involve only the positive-length asymptotic overlap and do not use the empty-word coefficient.

This rectangular dimension recovery is genuinely needed in the proof of Theorem `thm1` [CPGSV16, statement lines 1167–1170; proof line 1182]. There the relevant overlaps are per-block overlaps $\langle V^{(N)}(B_k) \,|\, V^{(N)}(A_j)\rangle$, and the source proof applies Lemma `equalMPS` before the matching of bond dimensions has been established. The same-dimension theorem therefore covers the gauge step only after the rectangular dimension conclusion has been supplied separately.

4 Verdict

The same-bond-dimension gauge component is formalized without the source-absent MPV proportionality hypothesis. The rectangular bond-dimension conclusion $D_a = D_b$ follows from the rectangular overlap-decay theorem through its non-decay contrapositive. The theorem `MPSTensor.IsNormalTensor.overlap_dichotomy` in `TNLean/MPS/Overlap/NormalTensorDichotomy.lean` now assembles these components with normalized self-overlap into the full normal-tensor dichotomy and structural conclusion of Lemma `equalMPS`. Its corollary `MPSTensor.IsNormalTensor.mpv_phase_alternative` proves the exact phase relation for every chain length, as stated in Corollary `eqV`. Thus the gap recorded by this note is closed; the preceding sections retain the mathematical decomposition used in the proof.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.