The marginal-support expansion in CPSV16 Appendix D.2

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Verdict: local-correction (resolved).

Let $\mathcal{H}_A$, $\mathcal{H}_X$, and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces, let $K_{AXB} \subseteq \mathcal{H}_A \otimes \mathcal{H}_X \otimes \mathcal{H}_B$, and let P_{AXB} be the orthogonal projector onto K_{AXB}. Define

$$\rho_{AX} = \operatorname{tr}_B P_{AXB}, \qquad \rho_{XB} = \operatorname{tr}_A P_{AXB}.$$

The two marginal supports are $\operatorname{ran} \rho_{AX}$ and $\operatorname{ran} \rho_{XB}$, with orthogonal projectors P_{AX} and P_{XB}, respectively. For an operator on AX or XB, a hat denotes its lift to the full tripartite space; in particular,

$$\widehat{P}_{AX} = P_{AX} \otimes I_B, \qquad \widehat{P}_{XB} = I_A \otimes P_{XB},$$

and likewise for $\widehat{\rho}_{AX}$ and $\widehat{\rho}_{XB}$.

1 The source step

In the proof of Proposition D.3, the source chooses orthonormal bases $\{\alpha_i\}$ and $\{\beta_i\}$ of the two marginal supports and asserts that every basis vector is a single partial contraction,

$$\alpha_i = \langle \widetilde{b}_i \mid \varphi_{b,i} \rangle, \qquad \beta_i = \langle \widetilde{a}_i \mid \varphi_{a,i} \rangle, \tag{1}$$

with $\varphi_{a,i}, \varphi_{b,i} \in K_{AXB}$. This is the passage at source lines 2236–2242.

The support of a reduced operator is spanned by partial contractions, but an arbitrary vector in that span need not itself be one partial contraction. Thus the displayed single-contraction choice does not follow from the definition of marginal support.

2 Local correction

The subsequent matrix-unit argument only needs a factorization of each marginal support projector through the corresponding reduced operator. For the elementary matrix units on the outer factors, write $E^A_{ak} = |a\rangle\langle k|$ and $E^B_{bk} = |b\rangle\langle k|$. The reduced operators have the exact expansions

$$\widehat{\rho}_{AX} = \sum_{b,k} E^B_{bk} P_{AXB} E^B_{kb}, \tag{2}$$

$$\widehat{\rho}_{XB} = \sum_{a,k} E^A_{ak} P_{AXB} E^A_{ka}. \tag{3}$$

If $g(0) = 0$ and $g(x) = x^{-1}$ for $x \neq 0$, functional calculus gives

$$P_{AX} = g(\rho_{AX})\rho_{AX} = \rho_{AX} g(\rho_{AX}), \tag{4}$$

$$P_{XB} = g(\rho_{XB})\rho_{XB} = \rho_{XB} g(\rho_{XB}). \tag{5}$$

Consequently the decorrelation identity first annihilates

$$\widehat{\rho}_{XB}(I - P_{AXB})\widehat{\rho}_{AX} \quad \text{and} \quad \widehat{\rho}_{AX}(I - P_{AXB})\widehat{\rho}_{XB}, \tag{6}$$

and multiplication by the two reciprocal-on-support factors gives the same equalities for $\widehat{P}_{XB}$ and $\widehat{P}_{AX}$. The absorption identities then yield

$$\widehat{P}_{XB}\widehat{P}_{AX} = P_{AXB} = \widehat{P}_{AX}\widehat{P}_{XB}. \tag{7}$$

This correction uses the source's matrix-unit and marginal-support argument, but replaces the unsupported single-contraction assertion by a valid factorization. No hypothesis is added to Proposition D.3.

Verdict: local correction. The conclusion and hypotheses of Proposition D.3 are unchanged; only the unsupported intermediate representation is replaced.

3 Formal statement

The reduced-operator expansions are `TripartiteDecorrelation.lift_marginalAX_eq_sum` and `TripartiteDecorrelation.lift_marginalXB_eq_sum`. The complementary support products are `TripartiteDecorrelation.IsDecorrelated. supports_mul_complement_mul_eq_zero` and its reverse-order counterpart. The two product identities are assembled by `TripartiteDecorrelation.supportProducts_eq`.

References