BNT Uniqueness and Zero Coefficients in CPSV16

2026-07-18

Verdict: false-source (resolved).

1 The source claim

Cirac–Pérez-García–Schuch–Verstraete define a basis of normal tensors for a tensor A to be a finite family of normal tensors $(A_j)_{j=1}^{g}$ such that, at each positive length, $V^{(N)}(A)$ is a linear combination of the $V^{(N)}(A_j)$, and these vectors are linearly independent for all sufficiently large N [CPGSV16, lines 271–274]. Proposition 2.7 characterizes such a family by coverage of the normal blocks of a canonical form and pairwise distinction up to a phase and an invertible gauge [CPGSV16, lines 278–280 and 1135–1146].

Immediately afterwards, the source asserts that any two bases of normal tensors corresponding to the same canonical form coincide up to phases and gauge transformations [CPGSV16, line 1148]. Read literally, this would in particular imply equality of the two cardinalities.

2 A two-symbol counterexample

Let the physical alphabet be $\{0, 1\}$ and let the bond dimension be one. Define two tensors by

$$A_0^0 = 1, \qquad A_0^1 = 0, \qquad A_1^0 = 0, \qquad A_1^1 = 1. \tag{1}$$

Both tensors are normal. Their transfer maps are the identity on the one-dimensional matrix algebra. Moreover,

$$V^{(N)}(A_0) = |0 \cdots 0\rangle, \qquad\qquad V^{(N)}(A_1) = |1 \cdots 1\rangle \tag{2}$$

for every $N \geq 1$. These two vectors are orthogonal and hence linearly independent.

The singleton family (A_0) is therefore a basis of normal tensors for A_0. The two-element family (A_0, A_1) is also a basis of normal tensors for A_0: take the coefficient of A_0 to be one and the coefficient of A_1 to be zero at every length. Both families correspond to the same one-block canonical-form display

$$V^{(N)}(A_0) = 1^N V^{(N)}(A_0). \tag{3}$$

Their index sets have cardinalities one and two, so no bijection, and hence no bijective gauge-phase matching, can exist.

The formal counterexample records both bases of normal tensors and the cardinality obstruction explicitly.[1]

3 Two restricted uniqueness statements

The obstruction is the identically zero coefficient. Coverage of the canonical-form blocks with nonzero coefficients gives a map from their gauge-phase classes into the candidate family. Pairwise distinction makes this map injective, but neither the literal definition nor Proposition 2.7 rules out a candidate that represents no such block. Thus the map need not be surjective.

A valid uniqueness theorem may be obtained by adding the converse coverage condition: every candidate must be gauge-phase equivalent to an active block of the displayed canonical form. Under this condition, the coverage maps are bijections and their composition gives the desired matching. This is a mathematically correct scope restriction, but it is not part of the literal hypotheses at lines 271–280 or 1135–1148. Accordingly, the unrestricted line-1148 sentence is not marked as formalized.

The converse-coverage restriction has been formalized: two such bases have equivalent index types, and the matched tensors have equal bond dimension and are related by an invertible gauge and a scalar of modulus one.[2] This restricted result does not alter the counterexample above or validate the unrestricted source sentence.

For the documented active canonical-block reading of Definition 4.7, a second restricted statement is used. An active BNT sector presentation consists of a nonempty family of normal representatives such that every representative has a positive number of copies, every copy weight is nonzero, the representative MPV states are eventually linearly independent, and the weighted direct sum has the same positive-length MPV family as the tensor being represented. Every representative is therefore an actual nonzero canonical summand of this direct sum.

Two active presentations of the same positive-length MPV family have a bijective gauge-phase matching. To prove this, apply [CPGSV16, Proposition 2.7] to the nonzero-weight direct-sum tensor of the first presentation, using the representatives of the second as a BNT, and then interchange the two presentations.

[1] The declaration is `MPSTensor.exists_isCPSVBasisOfNormalTensors_unequal_card` in `TNLean/MPS/CanonicalForm/BNTUniqueness.lean`.

[2] The declaration is `MPSTensor.IsCPSVBasisOfNormalTensors.equiv_of_converse_coverage` in `TNLean/MPS/CanonicalForm/BNTUniqueness.lean`.

Eventual linear independence excludes gauge-phase-equivalent distinct representatives. The resulting injections between the two finite index sets give the desired bijection.

This theorem is `MPSTensor.IsActiveCPSVBasisOfNormalTensors.equiv_of_sameMPVPos` in `TNLean/MPS/CanonicalForm/ActiveBNTRefinement.lean`. It is the scope restriction used for presentation independence of Definition 4.7. It does not formalize the unrestricted sentence at line 1148, and it does not impose the unit-modulus normalization at line 246.

4 The active-only consequence for Figure 11

The same distinction applies to the Figure 11 decomposition in Appendix C.4. For a fixed ordered pair (α, β), the source writes the active normal corners of the product tensor in the form

$$U_{\alpha,\beta}(M_\alpha M_\beta)^i U^\dagger_{\alpha,\beta} = \bigoplus_\gamma \chi_{\alpha,\beta,\gamma} \otimes Z_{\alpha,\beta,\gamma} M^i_\gamma Z^{-1}_{\alpha,\beta,\gamma} \tag{4}$$

[CPGSV16, Appendix C.4, lines 2020–2029]. The index γ in this display ranges over the normal sectors that occur with a nonzero multiplicity for the chosen pair. BNT uniqueness does not imply that every candidate normal tensor occurs for every pair.

In particular, the fiber of $\chi_{\alpha,\beta,\gamma}$ may be empty. Such a fiber has no positive diagonal entry and contributes zero to every positive-length power sum. If the whole product tensor $M_\alpha M_\beta$ vanishes, all its χ fibers are empty and the right-hand side of (4) is the empty direct sum. Adding an unrelated normal tensor with coefficient zero would again satisfy the weak spanning clause, but it would not turn that tensor into an active Figure 11 corner.

Thus the corresponding consequence of BNT comparison is active-only: every nonzero normal corner of a fixed product pair is gauge-phase equivalent to a normal sector M_γ, while absent sectors are omitted rather than represented by zero coefficients. Equality of the global closed-chain sums over all (α, β) cannot strengthen this statement to nonempty support for each fixed pair. The tensor-level reason and the treatment of zero pairs are recorded in `docs/paper-gaps/cpsv16_figure11_per_pair_support.tex`.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.