

Historical Note: Rate-Quantified BNT Cross-Overlap Decay for a Retired Projection Argument

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Verdict: unfaithful (historical).

1 Status

Historical proof archaeology, not an active gap. The corrected proportional fundamental theorem is `MPSTensor.fundamentalTheorem_proportional_canonicalForm`. It compares active BNT families whose listed copy weights are nonzero and obtains a bijection of the basis sectors with gauge-phase equivalences. The unrestricted statement printed in CPSV16 remains false: its weak BNT convention allows an unrelated normal representative whose coefficient is identically zero, so the claimed equality of BNT cardinalities fails. No rate-quantified overlap estimate is needed for the corrected theorem. The analysis below records the extra hypothesis required by the retired per-block projection attempt.

2 Notation

Fix a sector decomposition P with basis blocks $\{P.\text{basis } j\}_{j=0}^{g-1}$ and a spectral level $\lambda_j \in \mathbb{C} \setminus \{0\}$ with $|\lambda_0| > |\lambda_1| > \dots > |\lambda_{g-1}|$ and $|\lambda_0| = 1$. The bilinear cross-overlap between basis blocks j and k at length N is

$$O_{j,k}(N) := \text{mpvOverlap}(P.\text{basis } j, P.\text{basis } k)(N) = \sum_{\sigma} \text{mpv}(P.\text{basis } j)(\sigma) \overline{\text{mpv}(P.\text{basis } k)(\sigma)}, \quad (1)$$

written `mpvOverlap` in the formalization.¹

For every ordered pair $j \neq k$ we are interested in a geometric bound

$$|O_{j,k}(N)| \leq C_{j,k} \eta_{j,k}^N \quad (\forall N \in \mathbb{N}), \quad (2)$$

with $C_{j,k} \geq 0$ and a rate $\eta_{j,k} \in [0, 1)$.

¹Declaration `MPSTensor.mpvOverlap` in `TNLean/MPS/Overlap/Basic`.

3 Cited assertion

The Step 1 argument in the proof of Theorem `thm1` [CPGSV16, line 1182] relies on $O_{j,k_0}(N) \rightarrow 0$ for $j \neq k_0$ to discard cross-overlap terms when projecting an assembled BNT identity onto a distinguished basis block $V^{(N)}(P.\text{basis } k_0)$. The qualitative limit is implicit in the BNT structure: distinct normal blocks satisfy

$$O_{j,k}(N) \xrightarrow[N \rightarrow \infty]{} 0 \quad (j \neq k). \quad (3)$$

The argument as written does not quantify the rate of decay in (3).

Definition 4.2 of [CPGSV21] packages (3) indirectly under *eventual linear independence* of the basis MPVs and does not state a rate. The formalization records this eventual linear-independence input in the shared sector-decomposition infrastructure,² while (3) is provided pointwise by `MPSTensor.mpvOverlap_tendsto_zero_of_not_gaugePhaseEquiv_cast_left_of_irreducible_TP`.³ Neither source equips $\eta_{j,k}$ with a quantitative relation to the spectral weight ratios.

4 The retired rate hypothesis

An earlier formal route introduced an explicit rate hypothesis on a sector decomposition P and a chosen spectral level $\lambda : \text{Fin } P.\text{basisCount} \rightarrow \mathbb{C}$. That route has since been retired together with the per-block-projection layer. The mathematical hypothesis it tried to isolate was the following: there would have to exist non-negative families C, η such that for every $j \neq k$,

$$0 \leq C_{j,k}, \quad 0 \leq \eta_{j,k} < 1, \quad \eta_{j,k} \cdot |\lambda_j / \lambda_k| < 1, \quad (4)$$

together with the geometric bound (2) for all N .

The surviving `SectorBNT` surface does not package such constants. It keeps the raw BNT weights in `SectorDecomposition` and records qualitative consequences in `TNLean/MPS/FundamentalTheorem/SectorBNT/Api`: the raw power-sum coefficient identity `SectorDecomposition.coeff_eq_sum_weight_pow`, within-family cross-decay `MPSTensor.IsBNTCanonicalForm.cross_overlap_basis_tendsto_zero`, the combined-family linear-independence criterion `MPSTensor.IsBNTCanonicalForm.combined_family_eventually_li`, and the modulus-free non-vanishing coefficient statement `MPSTensor.IsBNTCanonicalForm.coeff_not_eventually_zero`. None of these declarations supplies the constants in (4).

²`TNLean/MPS/SharedInfra/SectorDecomposition`.

³`TNLean/MPS/Overlap/CastDecay`.

5 Where the retired hypothesis was used

The retired per-block-projection route carried an abstract hypothesis excluding asymptotic cancellation of the assembled overlap against a chosen projection block k_0 . The obstruction analysis in `docs/audits/2026-05-13_cpsv16_ft_path_beta_pr2_5_design.md` shows that for a non-dominant k_0 the discharge requires, after isolating $V^{(N)}(P.\text{basis } k_0)$ on the right and renormalising by $\lambda_{k_0}^N$, that every $j \neq k_0$ contribute a term whose modulus is bounded by

$$|O_{j,k_0}(N)| \cdot |\lambda_j/\lambda_{k_0}|^N \leq C_{j,k_0}(\eta_{j,k_0} \cdot |\lambda_j/\lambda_{k_0}|)^N. \quad (5)$$

The right-hand side of (5) converges to 0 exactly when (4) holds. The qualitative limit (3) alone provides no constants C_{j,k_0} and $\eta_{j,k_0} < 1$, and hence no geometric rate. Such a bound requires a separate finite-dimensional spectral-radius analysis of the mixed transfer operator. Even after obtaining a geometric rate, the comparison $\eta_{j,k_0} \cdot |\lambda_j/\lambda_{k_0}| < 1$ is an additional compatibility condition required by the retired argument.

6 Classification

Historical analytic obstruction. The rate condition (4) is a structural input beyond what [CPGSV16, §II] and [CPGSV21, Definition 4.2] make explicit. Obtaining a geometric rate requires a separate finite-dimensional spectral-radius input for the mixed transfer operator underlying `mpvOverlap_tendsto_zero_of_not_gaugePhaseEquiv_cast_left_of_irreducible_TP`. Moreover, the compatibility inequality $\eta_{j,k} \cdot |\lambda_j/\lambda_k| < 1$ is not implied by the BNT structure. The retired projection argument would therefore have needed this condition as an additional hypothesis or a separate analytic theorem.

The design memo `docs/audits/2026-05-13_cpsv16_ft_path_beta_pr2_5_design.md` records the abandoned spectral-gap proposal. The corrected `SectorBNT` theorem uses coefficient comparison and combined-family linear independence instead. This note documents why the rate-quantified projection route must not be revived as a purported obligation of the proportional fundamental theorem.

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.