The Trace-Ratio Exponent in the Blocked Vertical-Operator Formula

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Verdict: local-correction (resolved).

1 The two formulas in Appendix C.4

Appendix C.4 of [CPGSV16] compares the one-site and two-site vertical canonical forms. Its lines 2001–2008 pair their sectors and show that the corresponding representative tensors satisfy

$$A^{(2)}_{\sigma(\alpha)} = \frac{m_\alpha}{n_{\sigma(\alpha)}} V_\alpha A^{(1)}_\alpha V^*_\alpha, \tag{1}$$

up to the harmless identification of equal bond dimensions. Here $m_\alpha = \mathrm{tr}(\mu_\alpha)$ and $n_\beta = \mathrm{tr}(\nu_\beta)$ are the traces of the positive diagonal multiplicity matrices.

For a closed chain of length L, multiplying every tensor letter by a scalar c multiplies the resulting operator by c^L. Unitary conjugation and the identification of the bond indices do not change the closing trace. Thus (1) gives the coefficient

$$\left(\frac{m_\alpha}{n_{\sigma(\alpha)}}\right)^L \mathrm{tr}\left(\nu^L_{\sigma(\alpha)}\right). \tag{2}$$

Line 2013 instead prints

$$\frac{m^L_\alpha}{n_{\sigma(\alpha)}} \mathrm{tr}\left(\nu^L_{\sigma(\alpha)}\right). \tag{3}$$

The two expressions differ by the factor $n^{L-1}_{\sigma(\alpha)}$. No normalization setting this factor equal to one appears in the source.

2 Confirmation later in the source

The same appendix resolves the exponent unambiguously at line 2040, where it uses

$$\mathrm{tr}\left[\left(\frac{m_\alpha}{n_{\sigma(\alpha)}}\nu_{\sigma(\alpha)}\right)^L\right] = \left(\frac{m_\alpha}{n_{\sigma(\alpha)}}\right)^L \mathrm{tr}\left(\nu^L_{\sigma(\alpha)}\right). \tag{4}$$

This is precisely (2). It also agrees directly with the tensor-letter scaling at line 2008. Therefore the denominator in line 2013 is missing the exponent L; this is a local typographical error, not an additional hypothesis or a change in the mathematical argument.

3 Formal correction

The simultaneous blocked-operator representations use (2). The formal statement retains the same sector equivalence, bond-dimension identifications, and unitary matrices that give (1); it introduces no normalization of the multiplicity matrices. The corresponding declaration is `MPOTensor.transportedVerticalSector_exists_blockedOperatorRepresentations` in `TNLean/MPS/MPDO/VerticalBlockedOperatorRepresentations.lean`. This correction is tracked in GitHub issue #4586.

Verdict. The coefficient consistent with both the preceding tensor identity and the later trace-power calculation is $\left(m_\alpha/n_{\sigma(\alpha)}\right)^L \operatorname{tr}(\nu_{\sigma(\alpha)}^L)$. The expression printed at line 2013 has a missing exponent on the denominator.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.